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# Verification of a low-degree polynomial vanishing at empirical points ☆

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ABSTRACT

Given a set of distinct empirical points with uniform tolerance, based on the LDP algorithm proposed by Fassino and Torrente, we provide a verification algorithm that computes a polynomial, an admissible perturbed point set with verified error bound, such that the polynomial is guaranteed to vanish at a slightly admissible perturbed point set within computed error bound. The effectiveness of our algorithm is demonstrated in several examples.



## 1. Introduction

As we all know, an algebraic way to describe the characteristics of a set of points is the vanishing ideal that comprises all polynomials vanishing at the given points. The vanishing ideal is classically determined using the BM algorithm [2] [12] [13], which returns a Gröbner basis of this ideal. However, if the point set is an empirical point set representing real-world measurements, the coordinates of the points inevitably include errors. Indeed, even small perturbation of points may change the structure of the Gröbner basis of the vanishing ideal. Therefore, the BM algorithm is not suitable for computing the vanishing ideal of an empirical point set.

As a stepping stone of symbolic and numerical computation, Stetter's book [18] indicates that the border basis and the homogeneous Gröbner basis are more stable tools in the area of numerical polynomial algebra. Using the H-basis, which is a form of the homogeneous Gröbner basis, Sauer [17] provided an algorithm to compute a set of polynomials that generates an approximate vanishing ideal. Based on the singular value decomposition technique, Heldt et al. [9] provided the AVI algorithm for computing an approximate border basis of the approximate vanishing ideal. In [1], Abbott et al. presented the SOI algorithm for computing a stable order ideal, which supports a stable border basis of the approximate vanishing ideal. In [4], by analyzing the sensitivity of least squares problem, Fassino gave the NBM algorithm to computing a set of polynomials whose elements almost vanish at the empirical point set. In the aspect of testing the linear dependence of columns of the evaluation matrix, Fassino's method is superior to the singular value decomposition method. Additionally,

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Kreuzer et al. [11] reviewed the techniques about studying the numerical stability in the computation involving multivariate polynomials.

Among the polynomials in an approximate vanishing ideal of an empirical point set, a crucial role is played by the low-degree polynomial, which provides a simple geometrical representation of empirical points. Motivated by this, by solving an underdetermined nonlinear system subject to constraints, Fassino and Torrente [7] introduced the LDP algorithm for computing a simple polynomial whose affine variety nearly contains the set of empirical points. Based on the QR decomposition, Fassino and Möller [5] provided an algorithm for computing the polynomial of the best approximation on the set of empirical points. Recently, Fassino, Möller and Riccomagno [6] presented the LDP-LP algorithm to handle the complex situations of the problems mentioned above.

*Main contribution* Starting with a set of distinct empirical points, this paper attempts to verify a low-degree polynomial vanishing at an admissible perturbed point set. The algorithm of this paper consists of two stages. The first stage computes an admissible perturbed point set, a polynomial that vanishes at the admissible perturbed point set with high precision, or fails. When the first part does not fail, the second stage determines verified and narrow error bound of the computed perturbed point set in the first step. Furthermore, the algorithm tests if there exists an admissible perturbed point set within corresponding error bound such that the computed polynomial in the first step exactly vanishes at this admissible perturbed point set.

*Structure of the paper* Section 2 is a preparation of this paper. Section 3 provides the main results of this paper in detail. The corresponding algorithm is given in Section 4. Several examples are given to demonstrate the performance of our algorithm in Section 5.

## 2. Notation and preliminaries

In this section, we will define the key notation used throughout the paper and give some background results.

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the sets of nonnegative integers, real numbers, and positive real numbers, respectively. Let $\mathbb{R}[\boldsymbol{x}] := \mathbb{R}[x_1, x_2, \dots, x_n]$ be the polynomial ring in $n$ variables over the field $\mathbb{R}$. Let $\mathbb{T}(\boldsymbol{x}) := \mathbb{T}(x_1, x_2, \dots, x_n)$ be the set of all monomials in $\mathbb{R}[\boldsymbol{x}]$.

Let $\mathcal{O} = \{t_1, t_2, \dots, t_r\} \subset \mathbb{T}(\boldsymbol{x})$, and $\mathbb{X} = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_s\}$ with $\boldsymbol{p}_i = (p_{i,1}, p_{i,2}, \dots, p_{i,n})^T$ for $1 \le i \le s$ be a point set, then the $(i, j)$ entry of the matrix $M_{\mathcal{O}}(\mathbb{X})$ is defined by $t_j(\boldsymbol{p}_i)$.

If $M \in \mathbb{R}^{m\times n}$, then we refer to $M_{i,:}$ and $M_{:,i}$ as the $i$-th row and the $i$-th column of $M$, respectively. Suppose that the integers $i_1$, $i_2$, $j_1$, and $j_2$ satisfy $1 \le i_1 < i_2 \le m$ and $1 \le j_1 < j_2 \le n$. Then $M_{i_1:i_2,:}$ and $M_{:,j_1:j_2}$ respectively stand for the submatrices of $M$ formed by selecting from the $i_1$-th row to the $i_2$-th row and from the $j_1$-th column to the $j_2$-th column. Suppose $z$ is scalar, and $M(z)$ is an $m$ by $n$ matrix with entries $M_{i,j}(z)$. If $M_{i,j}(z)$ is a differentiable function of $z$ for all $i$ and $j$, then $M'(z)$ refers to the matrix with entries $M'_{i,j}(z)$. Besides, $I_n$ represents the $n$-by-$n$ identity matrix, and $O_{m,n}$ designates the $m$-by-$n$ zero matrix.

**Definition 1.** [12] A monomial set $\mathcal{O} \subset \mathbb{T}(\boldsymbol{x})$ is called an order ideal if it is closed under monomial division, namely, if $t_1 \in \mathcal{O}$ and $t_2 | t_1$, then $t_2 \in \mathcal{O}$.

**Definition 2.** [12] Let $t_1 := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $t_2 := x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}$.

1. $t_1 <_{\text{lex}} t_2$ if the leftmost nonzero entry of the vector $(\beta_1, \beta_2, \dots, \beta_n) - (\alpha_1, \alpha_2, \dots, \alpha_n)$ is positive.
2. Also, $t_1 <_{\text{grlex}} t_2$ if

$$\sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i, \quad \text{or} \quad \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i \text{ and } t_1 <_{\text{lex}} t_2.$$

**Definition 3.** [12] Given $f \in \mathbb{R}[\boldsymbol{x}]$, the affine variety of $f$ is defined by $\mathcal{V}(f) = \{\boldsymbol{p} \in \mathbb{R}^n : f(\boldsymbol{p}) = 0\}$.

**Definition 4.** [1] [7] Let $\boldsymbol{p} \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}^+$.

1. A pair $(\boldsymbol{p}, \varepsilon) = \boldsymbol{p}^{\varepsilon}$ is called an empirical point with the specified value $\boldsymbol{p}$ and tolerance $\varepsilon$.
2. The $\varepsilon$-neighborhood of $\boldsymbol{p}^{\varepsilon}$ is defined as

$$N(\boldsymbol{p}^{\varepsilon}) = \{\widetilde{\boldsymbol{p}} \in \mathbb{R}^n : \|\widetilde{\boldsymbol{p}} - \boldsymbol{p}\|_{\infty} \le \varepsilon\}.$$

3. A point $\widetilde{\boldsymbol{p}}$ is called an admissible perturbed point of $\boldsymbol{p}$ if $\widetilde{\boldsymbol{p}} \in N(\boldsymbol{p}^{\varepsilon})$.

**Definition 5.** [1] [7] Two empirical points $\boldsymbol{p}_1^{\varepsilon}$ and $\boldsymbol{p}_2^{\varepsilon}$ are said to be distinct if $N(\boldsymbol{p}_1^{\varepsilon}) \bigcap N(\boldsymbol{p}_2^{\varepsilon}) = \emptyset$.