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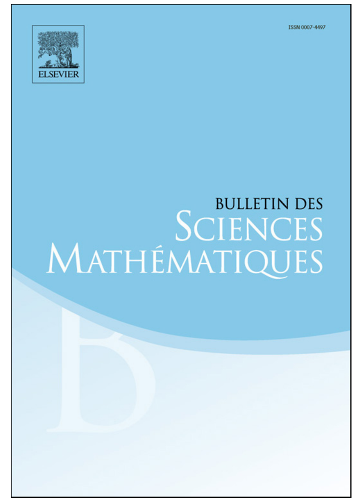

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CONJUGATION ORBITS OF LOXODROMIC PAIRS IN $\mathrm{SU}(n,1)$

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ABSTRACT. Let $\mathbf{H}_{\mathbb{C}}^n$ be the n-dimensional complex hyperbolic space and $\mathrm{SU}(n,1)$ be the (holomorphic) isometry group. An element g in $\mathrm{SU}(n,1)$ is called *loxodromic* or *hyperbolic* if it has exactly two fixed points on the boundary $\partial\mathbf{H}_{\mathbb{C}}^n$. We classify $\mathrm{SU}(n,1)$ conjugation orbits of pairs of loxodromic elements in $\mathrm{SU}(n,1)$.

1. Introduction

Let $\mathbf{H}_{\mathbb{C}}^n$ be the n-dimensional complex hyperbolic space. The group $\mathrm{SU}(n,1)$ acts by the holomorphic isometries on $\mathbf{H}_{\mathbb{C}}^n$. An element of $\mathrm{SU}(n,1)$ is called *hyperbolic* or *loxodromic* if it has exactly two fixed points on the boundary $\partial\mathbf{H}_{\mathbb{C}}^n$ of the complex hyperbolic space.

Let $\mathrm{F}_2 = \langle x, y\rangle$ be a two-generator free group. Let $\mathfrak{X}(\mathrm{F}_2, \mathrm{SU}(n,1))$ denote the orbit space $\mathrm{Hom}(\mathrm{F}_2, \mathrm{SU}(n,1))/\mathrm{SU}(n,1)$ of the conjugation action of $\mathrm{SU}(n,1)$ on the space $\mathrm{Hom}(\mathrm{F}_2, \mathrm{SU}(n,1))$ of faithful representations of F_2 into $\mathrm{SU}(n,1)$. Let $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$ denote the subset of $\mathfrak{X}(\mathrm{F}_2, \mathrm{SU}(n,1))$ consisting of representations ρ such that both $\rho(x)$ and $\rho(y)$ are loxodromic elements in $\mathrm{SU}(n,1)$ having no common fixed point. A problem of geometric interest is to parametrize this subset $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$. The motivation for doing this is the construction of Fenchel-Nielsen coordinates in the classical Teichmüller space that is built upon a parametrization of discrete, faithful, and totally loxodromic representations in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SL}(2,\mathbb{R}))$. This is rooted back to the classical works of Fricke [Fri96] and Vogt [Vog89] from whom it follows that a non-elementary two-generator free subgroup of $\mathrm{SL}(2,\mathbb{R})$ is determined up to conjugation by the traces of the generators and their product, see Goldman [Gol09] for an exposition.

The space $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$ contains the discrete, faithful, and totally loxodromic or type-preserving representations. These are curious families of representations and has not been well-understood even in the case $n = 2$. We refer to the surveys [PP10], [Sch02], [Wil16] and the references therein for an up to date account of the investigations in this direction.

For notational convenience, an element in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$ will be called a 'loxodromic generated representation', or simply, a 'loxodromic representation'. Most of the existing works to understand $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$ is centered around the case $n = 2$, though it would be interesting to generalize some of above mentioned works for $n > 2$. A starting point for this could be the classification of pairs of elements in $\mathrm{SU}(n,1)$ up to conjugacy. In other words, the problem would be to determine a representation in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$.

To do this, following classical invariant theory, one approach is to obtain this classification using trace invariants like the coefficients of the characteristic polynomials and their compositions. In low dimensions, this approach gives some understanding of the loxodromic pairs. Will [Wil06, Wil09] classified the loxodromic pairs in $\mathrm{SU}(2,1)$. Will's classification is built upon the work of Lawton [Law07], also see [Wen94], who obtained trace parameters for elements in $\mathfrak{X}(\mathrm{F}_2, \mathrm{SL}(3,\mathbb{C}))$. It follows from these works that an irreducible loxodromic representation in

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