Neurocomputing 000 (2018) 1–14

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Geometric measures of entanglement in multipartite pure states via complex-valued neural networks

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ARTICLE INFO

Article history:
Received 18 January 2018
Revised 11 May 2018
Accepted 27 May 2018
Available online xxx

Communicated by Prof. Qiankun Song

AMS subject classications:
15A18
15A69
65F15
65F10

Keywords:
Quantum entanglement
Multipartite state
Geometric measure of entanglement
Complex-valued neural network
Quantum eigenvalue problem
Local optimal complex rank-one approximation
Lyapunov stability theory
Complex symmetric tensors
Complex tensors

ABSTRACT

The geometric measure of entanglement of a multipartite pure state is defined it terms of its geometric distance from the set of separable pure states. The quantum eigenvalue problem is derived to compute the separable pure state nearest to the given multipartite pure state. Computing the modulus largest quantum eigenvalue for a multipartite pure state is equivalent to finding the best complex rank-one approximation of the complex unit tensors, associated with the multipartite pure states. This paper is devoted to present a complex-valued neural networks approach for the computation of the quantum eigenvalue problem for multipartite pure states. We design the neural networks for computing the best rank-one tensor approximation of complex tensors, and prove that the solution of the networks is locally asymptotically stable in the sense of Lyapunov stability theory. This solution also converges to the local optimal solutions of the best complex rank-one tensor approximation. We illustrate our theoretical results via numerical simulations.

1. Introduction

A tensor is an N-dimensional array of numbers denoted by script notation $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ with entries given by

$$a_{i_1 i_2 \ldots i_N} \in \mathbb{C}, \quad \text{for} \quad i_n = 1, 2, \ldots, I_n, \ \text{with} \ n = 1, 2, \ldots, N.$$

We use $CT_{N,I}$ to denote the set of order N dimension I complex tensors in general. That is, when $\mathcal{A} \in CT_{N,I}$, we have $a_{i_1 i_2 \ldots i_N} \in \mathbb{C}$ where $i_n = 1, 2, \ldots, I$ and $n = 1, 2, \ldots, N$.

The problem of best rank-one approximation of $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ is to find a real scalar σ and N unit vectors $\mathbf{x}_n \in \mathbb{C}^{I_n}$ ($\|\mathbf{x}_n\|_2 = 1$) that minimize

$$\sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \ldots \sum_{i_N=1}^{I_N} \left| a_{i_1 i_2 \ldots i_N} - \sigma \cdot (x_{1,i_1} x_{2,i_2} \ldots x_{N,i_N}) \right|^2,$$

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[1] This author is supported by the Fundamental Research Funds for the Central Universities under grant JBK1801058.
[2] This author's work was supported by the Hong Kong Research Grant Council (Grant No. PolyU 15302114, 15300715, 15301716 and 15300717).
[3] This author is supported by the National Natural Science Foundation of China under grant 11771099.
[4] This author is supported by the National Natural Science Foundation of China and the Hong Kong Research Grant Council (Grant No. 61374057 and 15206915).

https://doi.org/10.1016/j.neucom.2018.05.094
0925-2312/

where x_{n,i_n} is the i_nth element of $\mathbf{x}_n \in \mathbb{C}^{I_n}$ for $i_n = 1, 2, \ldots, I_n$ with $n = 1, 2, \ldots, N$, and $\sigma \in \mathbb{R}$ is a scaling factor. The relationship between the best rank-one approximation of complex tensors and geometric measures of entanglement in multipartite pure states will be discussed in Section 2.1.

There exist numerical methods to compute the best rank-one approximation of real tensors, e.g., the alternating least squares (ALS) method, truncated higher-order singular value decomposition, higher-order power method and semi-definite relaxations. We refer to Zhang and Golub [47], De Lathauwer, De Moor and Vandewalle [13,14], Kofidis and Regalia [27], Qi et al. [38], Ni and Wang [36], Nie and Wang [37] and the references therein.

Ni et al. [35] considered two eigenvalue problems of complex tensors: the U-eigenvalue problem of a complex tensor and the US-eigenvalue problem of a complex symmetric tensor, which are related to the best rank-one approximation of complex tensors. Recently, Ni and Bai [34] proposed an algorithm for computing the US-eigenpairs of complex symmetric tensors based on a spherical optimization problem of real-valued functions with complex variables. This algorithm was used to compute the upper bound of entanglement in an arbitrary multi-partite system [39]. Che et al. [9] presented iterative algorithms for computing US- (or U-) eigenpairs of complex tensors based on the Takagi factorization of complex matrices.

Wang et al. [44] proposed complex-valued neural network models for the computation of the Takagi vector of a complex symmetric matrix that corresponds to the largest Takagi values. The readers can refer to [2,3,24], which studied a complex nonlinear convex programming problem by means of complex-valued neural network models. Generally speaking, complex-valued neural networks have different and more complicated properties than real-valued ones. Thus, it is important to study the dynamical behaviors of complex-valued neural networks.

One important aspects of the dynamics of neural networks is their stability. To analyze the stability of neural networks, various approaches, such as Lyapunov function method and synthesis method, have been proposed [12,30,41]. Che et al. [8] presented a neural dynamical network to compute a local optimal rank-one approximation of a real tensor and proved that the state of the proposed neural network is locally asymptotically stable in the sense of Lyapunov stability theory. The main purpose of this paper is to design complex-valued neural network models for computing the local optimal rank-one approximation of complex tensors. We also derive that the solution of the complex-valued ODEs is locally asymptotically stable in the sense of Lyapunov stability theory. As shown in Section 7, the method of complex-valued neural network models is a strong tool for calculating geometric measure of entanglement.

Throughout this paper, we assume that *I*, *J*, and *N* will be reserved to denote the index upper bounds, unless stated otherwise. Scalars are denoted by lower Greek letters and lower Roman letters, e.g., α and a. Vectors are denoted by boldface letters and are lower case, e.g., $\mathbf{z}$. Matrices are denoted by block capital letters, e.g., $\mathbf{A}$. Tensors are denoted by calligraphic letters, e.g., $\mathcal{A}$. The superscripts $\cdot^\top$, $\bar{\cdot}$ and $\cdot^*$ are used for the transpose, the complex conjugate and conjugate transpose, respectively.

The two-norm and Frobenius norm are denoted by $\|\cdot\|_2$ and $\|\cdot\|_F$, respectively. The entry with row index i and column index j in a matrix $\mathbf{A}$, i.e., $(\mathbf{A})_{ij}$, is symbolized by a_{ij} (also $(\mathbf{z})_i = z_i$ and $(\mathcal{A})_{i_1 i_2 \ldots i_N} = a_{i_1 i_2 \ldots i_N}$). We use parentheses to denote the concatenation of two or more vectors, e.g., $(\mathbf{a}, \mathbf{b})$ is equivalent to $(\mathbf{a}^\top, \mathbf{b}^\top)^\top$. We use $\Re(\mathbf{z})$ and $\Im(\mathbf{z})$ to denote the real and imaginary parts of a vector $\mathbf{z} \in \mathbb{C}^I$.

The rest of this paper is organized as follows. In Section 2, we introduce basic notations about quantum states, convert the problem for measuring entanglement of a multipartite pure state to the complex best rank-one tensor approximation, and present the expressions for the complex gradient of real functions in complex variables. In Section 3, we define the generalized Rayleigh quotient of the complex tensors and establish the relationship between the local optimal complex rank-one tensor approximation and the nonlinear quantum eigenvalue problem (US-eigenvalue problems or U-eigenvalue problems [35]) based on the generalized Rayleigh quotient of any complex tensor. We present neural networks and consider the properties of these neural networks in Section 4. In Section 5, we establish the complex-valued neural networks to find the local optimal complex rank-one tensor approximation and analyze its local asymptotic stability in the sense of Lyapunov stability theory. We illustrate our theory via numerical simulations in Section 6 and conclude our paper in Section 7.

2. Preliminaries

The mode-n product [28] of a complex tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ by a matrix $\mathbf{B} \in \mathbb{C}^{J_n \times I_n}$, denoted by $\mathcal{A} \times_n \mathbf{B}$, is a tensor $\mathcal{C} \in \mathbb{C}^{I_1 \times \ldots \times I_{n-1} \times J_n \times I_{n+1} \times \ldots \times I_N}$, whose entries are given by

$$c_{i_1 \ldots i_{n-1} j i_{n+1} \ldots i_N} = \sum_{i_n=1}^{I_n} a_{i_1 i_2 \ldots i_N} b_{j i_n}, \quad n = 1, 2, \ldots, N.$$

In particular, the mode-n multiplication of a complex tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ by a vector $\mathbf{z} \in \mathbb{C}^{I_n}$ is denoted by $\mathcal{A} \times_n \mathbf{z}^\top$. If we set $\mathcal{C} = \mathcal{A} \times_n \mathbf{z}^\top \in \mathbb{C}^{I_1 \times \ldots \times I_{n-1} \times I_{n+1} \times \ldots \times I_N}$, then we have element-wise [28],

$$c_{i_1 \ldots i_{n-1} i_{n+1} \ldots i_N} = \sum_{i_n=1}^{I_n} a_{i_1 \ldots i_{n-1} i_n i_{n+1} \ldots i_N} x_{i_n}.$$

Given N vectors $\mathbf{z}_n \in \mathbb{C}^{I_n}$ $(n = 1, 2, \ldots, N)$, the notation $\mathcal{A} \times_1 \mathbf{z}_1^\top \times_2 \mathbf{z}_2^\top \ldots \times_N \mathbf{z}_N^\top$ is easy to define. For any given tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ and the matrices $\mathbf{F} \in \mathbb{C}^{J_n \times I_n}$ and $\mathbf{G} \in \mathbb{C}^{J_m \times I_m}$, one has [28]

$$\begin{cases} (\mathcal{A} \times_n \mathbf{F}) \times_m \mathbf{G} = (\mathcal{A} \times_m \mathbf{G}) \times_n \mathbf{F} = \mathcal{A} \times_n \mathbf{F} \times_m \mathbf{G}; \\ (\mathcal{A} \times_n \mathbf{F}) \times_n \mathbf{G} = \mathcal{A} \times_n (\mathbf{G} \cdot \mathbf{F}), \quad \text{with} \quad J_n = I_m, \end{cases}$$

with $m \neq n \in \{1, 2, \ldots, N\}$, where '$\cdot$' represents the multiplication of two matrices.

If the entries of $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ are given by $a_{i_1 i_2 \ldots i_N} = x_{1,i_1} x_{2,i_2} \ldots x_{N,i_N}$, where x_{n,i_n} is the i_nth element of $\mathbf{x}_n \in \mathbb{C}^{I_n}$ for $n = 1, 2, \ldots, N$, then we call $\mathcal{A}$ a complex rank-one tensor [14,47].

2.1. Geometric measure of entanglement

Entanglement has been identified as a resource central to quantum information processing. As a result, the task of characterizing and quantifying entanglement is vitally important in quantum information theory. The geometric measure of entanglement is one of most natural and important measures for pure states in bipartite and multipartite systems. We refer to [42,45] and their inferences therein. Mathematically speaking, the geometric measure of entanglement is nothing but the injective tensor norm [21], which appears in the theory of operator algebra [15]. The geometric measure of entanglement also has found wild applications in various different topics, such as many-body physics [31,33], entanglement witnesses [17,20] and the study of quantum channel capacities [7,16,46].

Wei and Goldbart [45] extended the geometric measure of entanglement from a bipartite pure state [42] to a multipartite pure state via the entanglement eigenvalue of a nonlinear quantum eigenvalue problem. Ni et al. [35] studied the nonlinear quantum eigenvalue problem in two forms: the U-eigenvalue problem of a