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Partition of unity extension of functions on complex domains

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A R T I C L E I N F O

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A B S T R A C T

We introduce an efficient algorithm, called partition of unity extension or PUX, to construct an extension of desired regularity of a function given on a complex multiply connected domain in 2D. Function extension plays a fundamental role in extending the applicability of boundary integral methods to inhomogeneous partial differential equations with embedded domain techniques. Overlapping partitions are placed along the boundaries, and a local extension of the function is computed on each patch using smooth radial basis functions; a trivially parallel process. A partition of unity method blends the local extrapolations into a global one, where weight functions impose compact support. The regularity of the extended function can be controlled by the construction of the partition of unity function. We evaluate the performance of the PUX method in the context of solving the Poisson equation on multiply connected domains using a boundary integral method and a spectral solver. With a suitable choice of parameters the error converges as a tenth order method down to 10^{-14}.

1. Introduction

This paper addresses the issue of how to numerically construct an extension of a function defined on a complex domain. Without prior acquaintance with the topic it could appear simple. However, adding requirements on the global regularity of the extended function it becomes a non-trivial task. Furthermore, it is often desirable that the extended function has compact support and that it can be efficiently constructed. One important application for function extension is to extend the applicability of integral equation methods for solving partial differential equations (PDEs). Integral equation methods have been shown to be both highly accurate and efficient when solving homogeneous constant coefficient elliptic partial differential equations in complex geometry. Function extension is a key component in a framework for solving non-homogeneous elliptic PDEs, and furthermore to solve time-dependent equations such as the heat equation and extending from the solution of Stokes equations to Navier–Stokes equations [1–3].

The idea is to avoid solving the full linear elliptic inhomogeneous PDE, given on a complex domain, by splitting the problem into two. The right-hand side is extended to a geometrically simpler domain, such as a box, and as part of the full solution, a particular solution is computed on this simpler domain. The homogeneous problem is solved on the original domain with modified boundary conditions, such that the total solution is the sum of the two. This general idea has been used also for other numerical methods and this group of methods is often referred to as embedded boundary techniques. For simple geometries an arsenal of powerful solution methods are available, but their accuracy is often limited by the

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global regularity of the extended function. Several different function extension methods, used in this context, have been suggested in recent years [4–9].

The approach to function extension suggested by Askham et al. [4] is based on solving an harmonic equation. Given f on a domain $\bar{\Omega}$, use the values of f on the boundary of Ω as the Dirichlet data for the external Laplace problem

$$\Delta w = 0 \text{ in } \mathbb{R}^2 \setminus \Omega, \quad (1)$$

$$w = f \text{ on } \partial\Omega. \quad (2)$$

Then a globally continuous extension f^e of f is given by

$$f^e(\mathbf{y}) = \begin{cases} f(\mathbf{y}), & \mathbf{y} \in \Omega, \\ w(\mathbf{y}), & \mathbf{y} \in \mathbb{R}^2 \setminus \Omega. \end{cases} \quad (3)$$

This problem is solved by an integral equation based method and the extension will be in C^0, but will not have compact support. To obtain higher regularity the biharmonic equation can be solved instead of (1)–(2), or even polyharmonic.

In [6] by Stein et al. the unknown solution, instead of the forcing f, is extended and the extended problem is solved via an *immersed boundary system of equations*. The extension is defined as the solution to some high order PDE, which in a sense is similar to the method presented by Askham et al. Their respective work show the complexity of the problem and what price is considered reasonable to pay to obtain an extension. Moreover both observe that the accuracy of the solution to the associated PDE relies heavily on the regularity of said extension.

Another alternative, given by [9], is to extend the solution to the PDE, instead of the right hand side, from a previous timestep or iterate. This extension is used in a *penalty term* to approximately enforce Dirichlet boundary conditions and to force the solution to be the extension of the solution from the previous timestep in some penalty region. A function extension of global regularity k is created by matching normal derivatives of degree k of the given boundary data. The extension is expressed in a basis that is rapidly decaying with the distance to the boundary. Function extension can also be achieved by Fourier continuation methods: in 1D the domain of interest is embedded into a larger one and a smooth periodic extension is constructed, which yields an appropriate setting for spectral methods. Dimensional splitting is used for higher dimensional problems. See [7,8,10,5] and the references therein. The methods and the associated references included above is by no means a complete list of methods for function extension. In all mentioned cases above, no higher than a fourth order method is obtained for solving the Poisson equation.

In this paper, we present a new method, *Partition of Unity Extension*, or PUX, to compute a compactly supported extension of a function. We assume that the values of a function f are known at all points of a regular grid that fall inside a domain Ω, and we want to compute the values of the extended function on this regular grid outside of Ω. The domain Ω can be multiply connected. In the PUX method, overlapping circular partitions or patches are placed along the boundaries such that each is intersected by the boundary $\partial\Omega$ and a local extension is defined on each patch. A second layer of patches is placed outside of the first, on which the local values are defined to be zero. These zero patches enter the definition of the partition of unity function that is used to blend the local extensions into a global one, imposing compact support and regular decay to zero. The choice of functions used to build up the partition of unity function determines the regularity of the extended function.

The local extensions on the patches intersected by $\partial\Omega$ are determined using radial basis functions (RBFs). RBF centres are placed irregularly with the same distribution for each circular patch, and an RBF interpolant is determined via a least squares problem, using the values of f on the regular points inside Ω. The values of the local extension are then computed on the regular points inside the patch that fall outside of Ω. Always centring the patches at grid points of the regular grid, a matrix A can be precomputed once and be used for all patches. For each patch, an identification is made of which points are inside and outside Ω and a local least squares problem with the relevant rows of A is solved with the inside data, a trivially parallel task.

To assess the quality of a function extension it must be considered in its context of use, as there is no unique extension over the boundary of a domain. In this paper, we use it to solve the Poisson equation with an integral equation approach. Thus the results by Askham et al. in [4] are suitable for comparison.

The paper is organised as follows: in section 2 we detail how we solve the Poisson equation assuming an extension of the right hand side f is known. In section 3 we introduce the concepts and techniques from RBF interpolation and the partition of unity method that we need to introduce our method. The PUX method is presented in section 4, where a function extension is constructed. Thereafter follows section 5 with a discussion of sources of errors associated with function extension and solving the Poisson equation. Section 6 is a summary, combined with implementation details, for solving the Poisson equation with the techniques described in this paper. In section 7 we perform numerical experiments and carefully discuss parameter choices. Finally our conclusions and an outlook are presented in section 8.

2. The Poisson equation on irregular domains in a boundary integral method environment

To understand why a function extension is useful for solving linear elliptic PDEs, and why its construction is motivated to pursue, we sketch the solution procedure. Consider the Poisson equation with Dirichlet boundary conditions, stated as