Journal of Computational Physics 375 (2018) 240–262

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A mixed mimetic spectral element model of the rotating shallow water equations on the cubed sphere

D. Lee [a,b,*], A. Palha [c]

[a] Department of Mechanical and Aerospace Engineering, Monash University, Melbourne 3800, Australia
[b] Computer, Computational and Statistical Sciences, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
[c] Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, 5600 MB Eindhoven, the Netherlands

ARTICLE INFO

Article history:
Received 14 March 2018
Received in revised form 24 August 2018
Accepted 24 August 2018
Available online 28 August 2018

Keywords:
Mimetic
Spectral convergence
Shallow water
Cubed sphere

ABSTRACT

In a previous article [*J. Comp. Phys.* **357** (2018) 282–304] [4], the mixed mimetic spectral element method was used to solve the rotating shallow water equations in an idealized geometry. Here the method is extended to a smoothly varying, non-affine, cubed sphere geometry. The differential operators are encoded topologically via incidence matrices due to the use of spectral element edge functions to construct tensor product solution spaces in $H(\text{rot})$, $H(\text{div})$ and L_2. These incidence matrices commute with respect to the metric terms in order to ensure that the mimetic properties are preserved independent of the geometry. This ensures conservation of mass, vorticity and energy for the rotating shallow water equations using inexact quadrature on the cubed sphere. The spectral convergence of errors are similarly preserved on the cubed sphere, with the generalized Piola transformation used to construct the metric terms for the physical field quantities.

Published by Elsevier Inc.

1. Introduction

In recent years there has been much attention given to the use of mimetic or compatible finite element methods for the modelling of geophysical flows. This work has been motivated by the desire to preserve conservation laws in order to mitigate against biases in the solution over long time integrations [1]. These mimetic methods are designed to preserve the divergence and circulation theorems in the discrete form, as well as the annihilation of the gradient by the curl and the curl by the divergence. When appropriate solution spaces are chosen for the divergent, vector and rotational moments, this allows for the conservation of first (mass, vorticity) and higher (energy and potential enstrophy) moments in the discrete form [2–4]. Various classes of element types have been explored for this purpose, including Raviart–Thomas, Brezzi–Douglas–Marini and Brezzi–Douglas–Fortin–Marini elements [3,5–7]. Mimetic properties may also be recovered for standard collocated A-grid spectral elements [8] and primal/dual finite volume formulations [9].

When implemented on non-affine geometries, the convergence of errors may degrade for compatible finite element methods [10], due to the reduced order of the polynomials when scaled by non-constant metric terms. Several methods have been shown to rehabilitate the optimal convergence of Raviart–Thomas elements for the L_2 function space [7,11,12] by modifying how the metric terms are incorporated into the differential operators, however it is unclear if and how these methods are applicable to other families of compatible finite element methods.

* Corresponding author at: Computer, Computational and Statistical Sciences, Los Alamos National Laboratory, Los Alamos, NM 87545, USA.
E-mail address: davidr.lee@monash.edu (D. Lee).

https://doi.org/10.1016/j.jcp.2018.08.042
0021-9991/Published by Elsevier Inc.

In the present article we extend previous work on the use of mixed mimetic spectral elements for geophysical flows [4], hereafter LPG18, to a non-affine cubed sphere geometry. The method uses the spectral element edge functions [13], which are specified to satisfy the Kronecker delta property with respect to their integrals between nodes, so as to exactly satisfy the fundamental theorem of calculus with respect to the standard nodal spectral element basis functions. Combinations of standard nodal and edge functions are then used to construct tensor product solution spaces in higher dimensions for which the differential operators may be defined in a purely topological manner via the use of incidence matrices [14–16]. These incidence matrices allow for the preservation of the divergence and circulation theorems, as well as the annihilation of the gradient by the curl and the curl by the divergence in the discrete form. The incidence matrices also commute with the metric transformations between computational and physical space, such that both the mimetic properties and the spectral convergence of errors are preserved on smoothly varying, non-affine geometries [14,16]. Indeed, for the spectral mimetic least squares method, optimal convergence has also been demonstrated for irregular meshes that do not vary smoothly or converge to an affine geometry [17].

In LPG18 the conservation and convergence properties of the mixed mimetic spectral element method for rotating shallow water flows were demonstrated both theoretically through formal proofs in the discrete form, and experimentally, through numerical experiments on idealized doubly periodic geometries. Here we extend these results to a non-affine cubed sphere geometry via the use of the generalized Piola transformation [6,18,19]. This demonstrates that both the conservation laws derived from the mimetic properties, and the spectral convergence of errors, are preserved for the smoothly varying, non-affine mesh of the cubed sphere without the need to rehabilitate the method through the modification of the discrete differential operators.

The remainder of this article proceeds as follows. In Section 2 the formulation of the mixed mimetic spectral element method will be briefly discussed. Section 3 will discuss the formulation of the metric terms and their commuting properties with respect to the differential operators. The solution of the rotating shallow water equations on the cubed sphere using mixed mimetic spectral elements will be discussed in Section 4. Section 5 will present results from some standard test cases demonstrating the preservation of optimal spectral convergence and conservation laws on the cubed sphere, and finally Section 6 will discuss the conclusions of this work and some future directions we intend to pursue with this research.

2. Mixed mimetic spectral elements

In this section we introduce the construction of the mixed mimetic spectral element method. For a more detailed discussion see LPG18, as well as previous work [13–16] and references therein.

2.1. One dimensional nodal and histopolant polynomials

The mixed mimetic spectral element method is built off two types of one-dimensional polynomials: one associated with nodal interpolation, and the other with integral interpolation (histopolation) [13,20]. Subsequently, these two types of polynomials will be combined to generate the family of two-dimensional polynomial basis functions used to discretize the system.

Consider the canonical interval $I = [-1, 1] \subset \mathbb{R}$ and the Legendre polynomials, $L_p(\xi)$ of degree p with $\xi \in I$. The $p+1$ roots, ξ_i, of the polynomial $(1-\xi^2)\frac{\mathrm{d}L_p}{\mathrm{d}\xi}$ are called Gauss–Lobatto–Legendre (GLL) nodes and satisfy $-1 = \xi_0 < \xi_1 < \cdots < \xi_{p-1} < \xi_p = 1$. Let $l_i^p(\xi)$ be the Lagrange polynomial of degree p through the GLL nodes, such that

$$l_i^p(\xi_j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad i, j = 0, \ldots, p\,. \tag{1}$$

The explicit form of these Lagrange polynomials is given by

$$l_i^p(\xi) = \prod_{\substack{k=0 \\ k \neq i}}^{p} \frac{\xi - \xi_k}{\xi_i - \xi_k}\,. \tag{2}$$

Let $q_h(\xi)$ be a polynomial of degree p defined on $I = [-1, 1]$ and $q_i = q_h(\xi_i)$, then the expansion of $q_h(\xi)$ in terms of Lagrange polynomials is given by

$$q_h(\xi) := \sum_{i=0}^{p} q_i l_i^p(\xi)\,. \tag{3}$$

Because the expansion coefficients in (3) are given by the value of q_h in the nodes ξ_i, we refer to this interpolation as a *nodal interpolation* and we will denote the Lagrange polynomials in (2) by *nodal polynomials*. Using the nodal polynomials we can define another set of basis polynomials, $e_i^p(\xi)$, as