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Universal image systems for non-periodic and periodic Stokes flows above a no-slip wall

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ABSTRACT

It is well-known that by placing judiciously chosen image point forces and doublets to the Stokeslet above a flat wall, the no-slip boundary condition can be conveniently imposed on the wall Blake (1971) [8]. However, to further impose periodic boundary conditions on directions parallel to the wall usually involves tedious derivations because single or double periodicity in Stokes flow may require the periodic unit to have no net force, which is not satisfied by the well-known image system. In this work we present a force-neutral image system. This neutrality allows us to represent the Stokes image system in a universal formulation for non-periodic, singly periodic and doubly periodic geometries. This formulation enables the black-box style usage of fast kernel summation methods. We demonstrate the efficiency and accuracy of this new image method with the periodic kernel independent fast multipole method in both non-periodic and periodic geometries. We then extend this new image system to other widely used Stokes fundamental solutions, including the Laplacian of the Stokeslet and the Rotne–Prager–Yamakawa tensor.

1. Introduction

No-slip boundaries in Stokes flow are central to much flow phenomena. For example, for Brownian suspensions above a no-slip wall, the wall not only constrains the motion of particles, but fundamentally changes the self and collective Brownian motion of suspensions by inducing anisotropy and screening effects in the mobility of particles [1–3]. Another example is that swimming microorganisms may swim upstream near a no-slip boundary in an imposed flow due to either hydrodynamic or non-hydrodynamic causes [4–7].

To compute the Stokes flow above a no-slip wall, the image method of Blake [8] is a popular choice. For a Stokeslet above a wall Blake showed that the no-slip condition was satisfied by adding an image Stokeslet, a modified source doublet, and a modified force doublet to the original Stokeslet. Similar methods have also been developed by Mitchell and Spagnolie [9,10]. Recently, Gimbutas et al. [11] developed a simpler image system. This system invokes standard Stokes and Laplace kernel evaluations only, which is compatible with the Fast Multipole Method (FMM). However, to further impose periodic boundary conditions on the two directions parallel to the no-slip wall is no simple task, because different kernel summations in the image system need to be periodized simultaneously and coupled to each other. Nguyen and Leiderman [12] recently derived the Ewald summation formulation for the doubly periodic Stokeslet image system, but their method showed a non-optimal

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$\mathcal{O}(N^2)$ scaling for N point forces. Their method was recently applied in the study of ciliary beating [13]. To our knowledge, the singly periodic Stokeslet image system above a no-slip wall has not yet been derived.

For convenience and efficiency, it is desirable to develop an image system where each kernel sum $\boldsymbol{g}^t = \sum_s \boldsymbol{K}(\boldsymbol{x}^t, \boldsymbol{y}^s)\boldsymbol{q}(\boldsymbol{y}^s)$ can be independently computed and periodized. Here $\boldsymbol{x}^t$ and $\boldsymbol{y}^s$ are target and source points with indices s and t. $\boldsymbol{K}$ is the kernel function. The kernel sum could be simply written as $\boldsymbol{g} = \boldsymbol{K}\boldsymbol{q}$, where the indices s, t are suppressed. For Stokes and Laplace kernel sums, recently developed optimal fast periodic kernel summation methods with flexible periodic boundary conditions can be used, including the Spectral Ewald methods [14–16], which scale as $\mathcal{O}(N \log N)$, and the periodic Kernel Independent Fast Multipole Method (KIFMM) method by Yan and Shelley [17], which scales as $\mathcal{O}(N)$. However the image systems developed by Gimbutas et al. [11] does not work in this framework, because the partially periodic (i.e., simply or doubly periodic) summations for the Stokeslet and the Laplace monopole kernel do not allow a net force or a net monopole in a periodic box, as otherwise the infinite periodic summations diverge. Unfortunately this requirement is not satisfied by the image system of [11].

In this work we propose a new image system for the Stokeslet, which satisfies the neutrality condition by rearranging the Stokeslet and Laplace kernel sums in the image system by Gimbutas et al. [11]. Therefore any singly or doubly periodic kernel summation method can be used as a black-box routine to periodize this new image system.

In Section 2 we briefly derive the new image system. Numerical results for Stokeslet above a no-slip wall with non-periodic and doubly periodic boundary conditions are presented in Section 3. In Section 4 we extend the new image to the Laplacian of Stokeslet and the widely used Rotne–Prager–Yamakawa tensor [18,19]. We conclude this work with a brief discussion about its coupling to fast summation methods, and its extension to other kernels.

2. Formulation

We first consider a point force $\boldsymbol{f} = (f_1, f_2, f_3)$ located at $\boldsymbol{y} = (y_1, y_2, y_3)$ above an infinite no-slip wall at the plane $x_3 = 0$. We define the image force $\boldsymbol{f}^I = (f_1, f_2, -f_3)$ located at $\boldsymbol{y}^I = (y_1, y_2, -y_3)$ below the wall. The complete image system to satisfy the no-slip condition on the wall is given by Gimbutas et al. [11] following the Papkovich–Neuber technique:

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{f} + \boldsymbol{J}(\boldsymbol{x}, \boldsymbol{y}^I)\left(-\boldsymbol{f}^I\right) - \boldsymbol{u}^C(\boldsymbol{x}), \tag{1a}$$

$$\boldsymbol{u}^C(\boldsymbol{x}) = x_3 \nabla_x \phi(\boldsymbol{x}) - \hat{\boldsymbol{x}}_3 \phi(\boldsymbol{x}), \quad \phi(\boldsymbol{x}) \stackrel{\text{def}}{=} G^S(\boldsymbol{x}, \boldsymbol{y}^I) f_3^I + \boldsymbol{G}^D(\boldsymbol{x}, \boldsymbol{y}^I)(y_3 \boldsymbol{f}^I), \tag{1b}$$

where $\hat{\boldsymbol{x}}_3$ is the unit vector in the x_3 direction. In this expression three kernels are involved: the Laplace monopole kernel $G^S(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}$, the Laplace dipole kernel $\boldsymbol{G}^D(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi}\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3} = \nabla_y G^S(\boldsymbol{x}, \boldsymbol{y})$, and the Stokeslet $\boldsymbol{J}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{8\pi}\left(\frac{\boldsymbol{I}}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right)$. We set the fluid viscosity to $\eta = 1$ in $\boldsymbol{J}$ for simplicity. It is clear that the net force is $\boldsymbol{f} + (-\boldsymbol{f}^I) = (0, 0, 2f_3) \neq 0$ in the Stokes kernel sum, and the net monopole is $f_3^I = -f_3 \neq 0$ in the Laplace monopole kernel sum. This forbids us to apply partially periodic kernel sum methods directly. The requirement of neutrality is straightforward to understand for Laplace kernels. For Stokeslet this depends on the particular periodic boundary condition. With triply periodic periodic boundary condition, the net force within a periodic box does not have to be zero because the net force can be balanced by the global pressure gradient [20]. However the net force must be zero with singly and doubly periodic boundary conditions, as demonstrated by Lindbo and Tornberg [14].

To remove the net force and net monopole, we convert the third component of Stokes force into a Laplace monopole kernel sum following the idea of Tornberg and Greengard [21]. This involves tedious algebraic manipulations and we only summarize the results here. The new image system splits the flow velocity into 4 independent parts $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^S + \boldsymbol{u}^D + \boldsymbol{u}^{L1} + \boldsymbol{u}^{L2}$, where each part is computed by one kernel sum. In the following, $\boldsymbol{f}_{xy} = (f_1, f_2, 0)$ denotes the x_1, x_2 components of the point force $\boldsymbol{f}$, parallel to the no-slip wall.

$$\boldsymbol{u}^S = \boldsymbol{J}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{f}_{xy} + \boldsymbol{J}(\boldsymbol{x}, \boldsymbol{y}^I)\left(-\boldsymbol{f}_{xy}\right), \tag{2a}$$

$$\boldsymbol{u}^D = \left(x_3 \nabla_x - \hat{\boldsymbol{x}}_3\right)\phi^D(\boldsymbol{x}), \quad \text{with } \phi^D \stackrel{\text{def}}{=} \boldsymbol{G}^D(\boldsymbol{x}, \boldsymbol{y}^I) \cdot y_3(-f_1, -f_2, f_3)^T \tag{2b}$$

$$\boldsymbol{u}^{L1} = -\frac{1}{2}\left(x_3 \nabla_x - \hat{\boldsymbol{x}}_3\right)\phi^S(\boldsymbol{x}), \quad \text{with } \phi^S(\boldsymbol{x}) \stackrel{\text{def}}{=} G^S(\boldsymbol{x}, \boldsymbol{y}) f_3 + G^S(\boldsymbol{x}, \boldsymbol{y}^I)(-f_3), \tag{2c}$$

$$\boldsymbol{u}^{L2} = \frac{1}{2}\nabla_x \phi^{SZ}(\boldsymbol{x}), \quad \text{with } \phi^{SZ}(\boldsymbol{x}) \stackrel{\text{def}}{=} \left[G^S(\boldsymbol{x}, \boldsymbol{y})(f_3 y_3) + G^S(\boldsymbol{x}, \boldsymbol{y}^I)(-f_3 y_3)\right]. \tag{2d}$$

$\boldsymbol{u}^S$ denotes the Stokes kernel sum, ϕ^D denotes the Laplace dipole sum, and ϕ^S, ϕ^{SZ} denote two Laplace monopole sums. The values and gradients of ϕ^D, ϕ^S and ϕ^{SZ} are computed at the target point $\boldsymbol{x}$. It is straightforward to verify that $\boldsymbol{u}(\boldsymbol{x})$ is equivalent to the original image system in Eq. (1). In this new image system, the Stokeslet sum and the two Laplace monopole sums are obviously neutral. The Laplace dipole sum is intrinsically neutral, because each dipole source is the asymptotic limit of zero distance between equal and opposite charges. Therefore, each of the 4 kernel sums can be separately periodized, and we claim this new image system is applicable for non-periodic, singly periodic and doubly periodic systems.