ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

stochastic processes and their applications

www.elsevier.com/locate/spa

On uniform closeness of local times of Markov chains and i.i.d. sequences

Diego F. de Bernardini, Christophe Gallesco, Serguei Popov*

Department of Statistics, Institute of Mathematics, Statistics and Scientific Computation, University of Campinas-UNICAMP, rua Sérgio Buarque de Holanda 651, 13083-859, Campinas SP, Brazil

Received 1 March 2017; received in revised form 27 September 2017; accepted 30 October 2017
Available online xxxx

Abstract

In this paper we consider the field of local times of a discrete-time Markov chain on a general state space, and obtain uniform (in time) upper bounds on the total variation distance between this field and the one of a sequence of n i.i.d. random variables with law given by the invariant measure of that Markov chain. The proof of this result uses a refinement of the soft local time method of Popov and Teixeira (2015).

Keywords: Occupation times; Soft local times; Decoupling; Empirical processes

1. Introduction

The purpose of this paper is to compare the field of local times of a discrete-time Markov process with the corresponding field of i.i.d. random variables distributed according to the stationary measure of this process, in total variation distance. Of course, local times (also called occupation times) of Markov processes is a very well studied subject. It is frequently possible to obtain a complete characterization of the law of this field in terms of some Gaussian random field or process, especially in continuous time (and space) setup. The reader is probably familiar with Ray–Knight theorems as well as Dynkin's and Eisenbaum's isomorphism theorems; cf. e.g. [12,14]. One should observe, however, that these theorems usually work in the case when

* Corresponding author.
E-mail addresses: bernardini@ime.unicamp.br (D.F. de Bernardini), gallesco@ime.unicamp.br (C. Gallesco), popov@ime.unicamp.br (S. Popov).

https://doi.org/10.1016/j.spa.2017.10.015
0304-4149/

Please cite this article in press as: D.F. de Bernardini, et al., On uniform closeness of local times of Markov chains and i.i.d. sequences, Stochastic Processes and their Applications (2017), https://doi.org/10.1016/j.spa.2017.10.015.

the underlying Markov process is reversible and/or symmetric in some sense, something we do not require in this paper.

To explain what we are doing here, let us start by considering the following example: let $(X_j)_{j\geq 1}$ be a Markov chain on the state space $\Sigma = \{0, 1\}$, with the following transition probabilities: $\mathbb{P}[X_{n+1} = k \mid X_n = k] = 1 - \mathbb{P}[X_{n+1} = 1 - k \mid X_n = k] = \frac{1}{2} + \varepsilon$ for $k = 0, 1$, where $\varepsilon \in (0, \frac{1}{2})$ is small. Clearly, by symmetry, $(\frac{1}{2}, \frac{1}{2})$ is the stationary distribution of this Markov chain. Next, let $(Y_j)_{j\geq 1}$ be a sequence of i.i.d. Bernoulli random variables with success probability $\frac{1}{2}$. What can we say about the distance in total variation between the laws of $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$? Note that the "naïve" way of trying to force the trajectories to be equal (given $X_1 = Y_1$, use the maximal coupling of X_2 and Y_2; if it happened that $X_2 = Y_2$, then try to couple X_3 and Y_3, and so on) works only up to $n = O(\varepsilon^{-1})$. Even though this method is probably not optimal, in this case it is easy to obtain that the total variation distance converges to 1 as $n \to \infty$. This is because of the following: consider the event

$$\Xi^Z = \Big\{\frac{1}{n}\sum_{j=1}^{n-1} \mathbb{1}_{\{Z_j = Z_{j+1}\}} > \frac{1}{2} + \frac{\varepsilon}{2}\Big\},$$

where Z is X or Y. Clearly, the random variables $\mathbb{1}_{\{Z_j = Z_{j+1}\}}$, $j \in \{1, \ldots, n-1\}$ are i.i.d. Bernoulli, with success probabilities $\frac{1}{2} + \varepsilon$ and $\frac{1}{2}$ for $Z = X$ and $Z = Y$ correspondingly. Therefore, if $n \gg \varepsilon^{-2}$, it is elementary to obtain that $\mathbb{P}[\Xi^X] \approx 1$ and $\mathbb{P}[\Xi^Y] \approx 0$, and so the total variation distance between the *trajectories* of X and Y is almost 1 in this case.

So, even in the case when the Markov chain gets quite close to the stationary distribution just in one step, usually it is not possible to couple its trajectory with an i.i.d. sequence, unless the length of the trajectory is relatively short. Assume, however, that we are not interested in the exact trajectory of X or Y, but rather, say, in the number of visits to 0 up to time n. That is, denote

$$L_n^Z(0) = \sum_{j=1}^{n} \mathbb{1}_{\{Z_j = 0\}}$$

for $Z = X$ or Y. Are $L_n^X(0)$ and $L_n^Y(0)$ close in total variation distance for *all* n?

Well, the random variable $L_n^Y(0)$ has the binomial distribution with parameters n and $\frac{1}{2}$, so it is approximately Normal with mean $\frac{n}{2}$ and standard deviation $\frac{\sqrt{n}}{2}$. As for $L_n^X(0)$, it is elementary to obtain that it is approximately Normal with mean $\frac{n}{2}$ and standard deviation $\sqrt{n}\big(\frac{1}{2} + O(\varepsilon)\big)$. Then, it is also elementary to obtain that the total variation distance between these two Normals is $O(\varepsilon)$, *uniformly* in n (indeed, that total variation distance equals the total variation distance between the Standard Normal and the centered Normal with variance $(1 + O(\varepsilon))^2$; that distance is easily verified to be of order ε). This *suggests* that the total variation distance between $L_n^X(0)$ and $L_n^Y(0)$ should be also of order ε uniformly in n. Observe, by the way, that the distribution of the local times of a two-state Markov chain can be explicitly written (cf. [2]), so one can obtain a rigorous proof of the last statement in a direct way, after some work.

Let us define the *local time* of a stochastic process Z at site x at time n as the number of visits to x up to time n:

$$L_n^Z(x) = \sum_{j=1}^{n} \mathbb{1}_{\{Z_j = x\}}$$

(sometimes we omit the upper index when it is clear which process we are considering). The above example shows that, if one is only interested in the local times of the Markov chain (and

Please cite this article in press as: D.F. de Bernardini, et al., On uniform closeness of local times of Markov chains and i.i.d. sequences, Stochastic Processes and their Applications (2017), https://doi.org/10.1016/j.spa.2017.10.015.