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Concentration for Poisson U-statistics: Subgraph counts in random geometric graphs☆

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Abstract

Concentration bounds for the probabilities $\mathbb{P}(N \geq M + r)$ and $\mathbb{P}(N \leq M - r)$ are proved, where M is a median or the expectation of a subgraph count N associated with a random geometric graph built over a Poisson process. The lower tail bounds have a Gaussian decay and the upper tail inequalities satisfy an optimality condition. A remarkable feature is that the underlying Poisson process can have a.s. infinitely many points.

The estimates for subgraph counts follow from tail inequalities for more general local Poisson U-statistics. These bounds are proved using recent general concentration results for Poisson U-statistics and techniques involving the convex distance for Poisson processes.
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1. Introduction

Random geometric graphs have been a very active area of research for some decades. The most basic model of these graphs is obtained by choosing a random set of vertices in $\mathbb{R}^d$ and connecting any two vertices by an edge whenever their distance does not exceed some fixed

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parameter $\rho > 0$. In the seminal work [12] by Gilbert, these graphs were suggested as a model for random communication networks. In consequence of the increasing relevance of real-world networks like social networks or wireless networks, variations of Gilbert's model have gained considerable attention in recent years, see e.g. [5,16,17,7,20,8]. For a detailed historical overview on the topic, we refer the reader to Penrose's fundamental monograph [21] on random geometric graphs.

For a given random geometric graph $\mathfrak{G}$ and a fixed connected graph H on k abstract vertices, the corresponding *subgraph count* N^H is the random variable that counts the number of occurrences of H as a subgraph of $\mathfrak{G}$. Note that only non-induced subgraphs are considered throughout the present work. The resulting class of random variables has been studied by many authors, see [21, Chapter 3] for a historical overview on related results.

The purpose of the present paper is to prove *concentration inequalities* for N^H, i.e. upper bounds on the probabilities $\mathbb{P}(N^H \geq M + r)$ and $\mathbb{P}(N^H \leq M - r)$, when the vertices of $\mathfrak{G}$ are given by the points of a Poisson point process. Our concentration results for subgraph counts are established using two novel approaches, the first one yielding estimates for the case where M is the expectation, the second one for the case where M is a median of N^H.

Theorem 1.1. *Let η be a Poisson point process in $\mathbb{R}^d$ with non-atomic intensity measure μ. Let H be a connected graph on $k \geq 2$ vertices. Consider the corresponding subgraph count $N^H = N$ in a random geometric graph associated to η. Assume that μ is such that almost surely $N < \infty$. Then all moments of N exist and for all $r \geq 0$,*

$$\mathbb{P}(N \geq \mathbb{E}N + r) \leq \exp\left(-\frac{((\mathbb{E}N + r)^{1/(2k)} - (\mathbb{E}N)^{1/(2k)})^2}{2k^2 c_d}\right),$$

$$\mathbb{P}(N \leq \mathbb{E}N - r) \leq \exp\left(-\frac{r^2}{2k\mathbb{V}N}\right),$$

$$\mathbb{P}(N > \mathbb{M}N + r) \leq 2\exp\left(-\frac{r^2}{4k^2 c_d (r + \mathbb{M}N)^{2-1/k}}\right),$$

$$\mathbb{P}(N < \mathbb{M}N - r) \leq 2\exp\left(-\frac{r^2}{4k^2 c_d (\mathbb{M}N)^{2-1/k}}\right),$$

where $c_d > 0$ is a constant that depends on H and d only, $\mathbb{E}N$ and $\mathbb{V}N$ denote the expectation and the variance of N, and $\mathbb{M}N$ is the smallest median of N.

One might think that the study of subgraph counts needs to be restricted to finite graphs to ensure that all occurring variables are finite. We stress that this is *not* the case. There are Poisson processes in $\mathbb{R}^d$, such that the associated random geometric graph has a.s. infinitely many vertices but still a.s. a finite number of edges. Similarly, one can also have a.s. infinitely many edges, but still a finite number of triangles. This phenomenon seems to be quite unexplored so far. In the context of concentration properties, a natural question is whether concentration inequalities also hold in these situations. We emphasize that Theorem 1.1 only requires $N < \infty$ almost surely and hence covers such cases, as opposed to previous results from [24,18] where only finite intensity measures are considered.

For $r \to \infty$, the asymptotic exponents in the upper tail bounds for both the expectation and the median are equal to $1/k$. This is actually best possible as will be pointed out later. Note also that the asymptotic exponent of the estimates in previous results from [24,18] is $1/k$ for both the upper and the lower tail which is compatible with our upper tail bounds but worse than our lower tail inequalities.

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