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Invariance principles for tempered fractionally integrated processes

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Abstract

We discuss invariance principles for autoregressive tempered fractionally integrated moving averages in α-stable $(1 < \alpha \le 2)$ i.i.d. innovations and related tempered linear processes with vanishing tempering parameter $\lim_{N\to\infty} \lambda/N = \lambda_*$. We show that the limit of the partial sums process takes a different form in the weakly tempered $(\lambda_* = 0)$, strongly tempered $(\lambda_* = \infty)$, and moderately tempered $(0 < \lambda_* < \infty)$ cases. These results are used to derive the limit distribution of the ordinary least squares estimate of AR(1) unit root with weakly, strongly, and moderately tempered moving average errors.

Keywords: Invariance principle; Tempered linear process; Autoregressive fractionally integrated moving average; Tempered fractional stable/Brownian motion; Tempered fractional unit root distribution

1. Introduction

The present paper discusses partial sums limits and invariance principles for tempered moving averages

$$X_{d,\lambda}(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} b_d(k) \zeta(t-k), \qquad t \in \mathbb{Z} \tag{1.1}$$

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with i.i.d. innovation process $\{\zeta(t)\}$ with coefficients $b_d(k)$ regularly varying at infinity as k^{d-1}, viz.

$$b_d(k) \sim \frac{c_d}{\Gamma(d)} k^{d-1}, \qquad k \to \infty, \qquad c_d \neq 0, \quad d \neq 0 \tag{1.2}$$

where $d \in \mathbb{R}$ is a real number, $d \neq -1, -2, \ldots$ and $\lambda > 0$ is tempering parameter. In addition to (1.2) we assume that

$$\sum_{k=0}^{\infty} k^j b_d(k) = 0, \quad 0 \le j \le [-d], \qquad -\infty < d < 0, \tag{1.3}$$

$$\sum_{k=0}^{\infty} |b_d(k)| < \infty, \quad c_0 := \sum_{k=0}^{\infty} b_d(k) \neq 0, \qquad d = 0 \tag{1.4}$$

Assumptions (1.3) and (1.4) are not necessary for the convergence of the series in (1.1) but are essential for the validity of the invariance principles. An important example of such processes is the two-parametric class ARTFIMA(0, d, λ, 0) of tempered fractionally integrated processes, generalizing the well-known ARFIMA(0, d, 0) class, written as

$$X_{d,\lambda}(t) = (1 - \mathrm{e}^{-\lambda} B)^{-d} \zeta(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} \omega_{-d}(k) \zeta(t-k), \qquad t \in \mathbb{Z} \tag{1.5}$$

with coefficients given by power expansion $(1 - \mathrm{e}^{-\lambda} z)^{-d} = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} \omega_{-d}(k) z^k$, $|z| < 1$, where $\omega_{-d}(k) := \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)}$ for $d \in \mathbb{R} \setminus \mathbb{N}_-$ and $Bx(t) = x(t-1)$ is the backward shift. Due to the presence of the exponential tempering factor $\mathrm{e}^{-\lambda k}$ the series in (1.1) and (1.5) absolutely converges a.s. and in L_p under general assumptions on the innovations, and defines a strictly stationary process. On the other hand, for $\lambda = 0$ the corresponding stationary processes in (1.1) and (1.5) exist under additional conditions on the parameter d. See Granger and Joyeux [13], Hosking [14], Brockwell and Davis [4], Kokoszka and Taqqu [16]. We also note (see e.g. [10], Ch. 3.2) that the (untempered) linear process $X_{d,0}$ of (1.1) with coefficients satisfying (1.2) for $0 < d < 1/2$ is said long memory, while (1.2) and (1.3) for $-1/2 < d < 0$ is termed negative memory and (1.4) short memory, respectively, parameter d usually referred to as memory parameter.

The model in (1.5) appeared in Giraitis et al. [8], which noted that for small $\lambda > 0$, $X_{d,\lambda}$ has a covariance function which resembles the covariance function of a long memory model for arbitrary large number of lags but eventually decays exponentially fast. [8] termed such behavior ‘semi long-memory’ and noted that it may have empirical relevance for modeling of financial returns. Giraitis et al. [9] propose the semi-long memory ARCH(∞) model as a contiguous alternative to (pure) hyperbolic and exponential decay which are often very hard to distinguish between in a finite sample. On the other side, Meerschaert et al. [22] effectively apply ARTFIMA(0, d, λ, 0) in (1.5) for modeling of turbulence in the Great Lakes region.

The present paper obtains limiting behavior of tempered linear processes in (1.1) with small tempering parameter $\lambda = \lambda_N \to 0$ tending to zero together with the sample size. The important statistic is the partial sums process

$$S_N^{d,\lambda}(t) := \sum_{k=1}^{[Nt]} X_{d,\lambda}(k), \qquad t \in [0, 1] \tag{1.6}$$

of $X_{d,\lambda}$ in (1.1) with i.i.d. innovations $\{\zeta(t)\}$ in the domain of attraction of α-stable law, $1 < \alpha \le 2$. Functional limit theorems for the partial sums process play a crucial role in the

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