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Path transformations for local times of one-dimensional diffusions

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Abstract

Let X be a regular one-dimensional transient diffusion and L^y be its local time at y. The stochastic differential equation (SDE) whose solution corresponds to the process X conditioned on $[L^y_\infty = a]$ for a given $a \geq 0$ is constructed and a new path decomposition result for transient diffusions is given. In the course of the construction *Bessel-type motions* as well as their SDE representations are studied. Moreover, the Engelbert–Schmidt theory for the weak solutions of one dimensional SDEs is extended to the case when the initial condition is an entrance boundary for the diffusion. This extension was necessary for the construction of the Bessel-type motion which played an essential part in the SDE representation of X conditioned on $[L^y_\infty = a]$.

Keywords: Regular one-dimensional diffusions; Local time; Engelbert–Schmidt theory of weak solutions; Bessel processes; Time reversal; Path decomposition of Markov processes

1. Introduction

Conditioning a given Markov process X is a well-studied subject which has become synonymous with the term *h-transform*. If one wants to condition the paths of X to stay in a certain set, the classical recipe consists of finding an appropriate excessive function h, defining the transition probabilities of the conditioned process via h, and constructing on the canonical space a Markov process X^h with these new transition probabilities using standard techniques. This procedure is called an h-transform and its origins go back to Doob and his study of boundary

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limits of Brownian motion [7,8]. If h is a minimal excessive function with a pole at y (see Section 11.4 of [6] for definitions), then X^h is the process X conditioned to converge to y and killed at its last exit from y. We refer the reader to Chapter 11 of [6] for an in-depth analysis of h-transforms and their connections with time reversal and last passage times.

If X is a regular transient diffusion taking values in some subset of $\mathbb{R}$, $h := u(\cdot, y)$ is a minimal excessive function for every y in its state space, where u is the potential density of X. Moreover, y is the unique pole of this excessive function. Thus, the preceding discussion suggests that this h-transform conditions X to converge to y and kills it at its last exit from y. For a thorough discussion of h-transforms for one-dimensional diffusions and the proofs of certain results that are considered to be folklore in the literature we refer the reader to a recent manuscript by Evans and Hening [10]. The recent works of Perkowski and Ruf [21] and Hening [12] also consider specific cases of conditioning for one-dimensional diffusions.

In this paper we are interested in conditioning a one-dimensional regular transient diffusion on the value of its local time at its lifetime. We assume that the diffusion cannot be killed in the interior of its state space. It is well-known that (X, L^y) is a two-dimensional Markov process, where L^y is the local time of X at level y. If we would like to apply an h-transform to achieve our conditioning, we need to find a minimal excessive function of the pair (X, L^y) with a suitable pole so that the local time of the X^h equals a given number, say, $a \geq 0$ at its lifetime. The problem with this approach is that it requires the knowledge of the potential density of the Markov pair (X, L^y), which is in general not easily obtained or characterised. Moreover, it will require a killing procedure.

We shall follow a different approach and construct the conditioned process as a weak solution to a stochastic differential equation (SDE) with a suitably chosen drift and an additional randomisation, which in essence is what deviates our approach from that of an h-transform. Moreover, there will not be any killing involved. On our way to constructing this SDE we will obtain the following contributions that are of significant interest in their own right:

- *An extension of the Engelbert–Schmidt theory for the weak solutions of one-dimensional SDEs.* The Engelbert–Schmidt theory constructs the weak solutions of one-dimensional SDEs as a time and scale change of a Brownian motion. This construction fails if the initial condition is an entrance boundary. We extend the theory when the initial condition is an entrance boundary by a time and scale transformation of a 3-dimensional Bessel process.
- *SDE representation for Bessel-type motions.* We obtain the SDE representation for the excursions of X away from a point conditioned to last forever.
- *A new path-decomposition result for transient diffusions.* We will show that we can obtain the transient diffusion by suitably pasting together a recurrent transformation, which is introduced in [5], and a Bessel-type motion. As such, this will give an alternative way of simulating one-dimensional diffusions.

Returning back to our main point of interest, that is, constructing an SDE whose solution coincides in law with X conditioned on L^y_∞, we shall next see in brief how the above contributions play a role in this construction. Since we are interested in obtaining an SDE for the conditioned process this obviously necessitates the original process, X, being a solution of an SDE. In Section 2 we impose the standard Engelbert–Schmidt conditions in order to ensure that X itself is the unique weak solution of an SDE up to a, possibly finite, exit time from its state space. Our aim is to construct an SDE – for which weak uniqueness holds – such that the