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A brief comment on Maxwell(/Newton)[-Huygens] spacetime

James Owen Weatherall

Department of Logic and Philosophy of Science, University of California, Irvine, CA 92697, USA

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ABSTRACT

I provide an alternative characterization of a “standard of rotation” in the context of classical spacetime structure that does not refer to any covariant derivative operator.

Following recent work by Simon Saunders (2013) and Eleanor Knox (2014), a flurry of papers have addressed the question of how to understand the geometry presupposed by Newtonian gravitational theory, particularly in light of Corollary VI to the Laws of Motion in Newton's *Principia* (Dewar, 2017; Teh, 2017; Wallace, 2016a, 2017; Weatherall, 2016c).[1] At issue has been the relationship between (1) Saunders' proposal that one can (and should) take the “correct” geometry for Newtonian gravitational theory to be that of what Earman (1989) called “Maxwellian spacetime”, and which more recently has been called “Newton-Huygens spacetime” (Saunders, 2013) or “Maxwell-Huygens spacetime” (Weatherall, 2016c), and (2) Knox's proposal that Corollary VI should motivate a move to geometrized Newtonian gravitation (i.e., Newton-Cartan theory).

One (somewhat tangential) thread of this discussion has concerned how to best characterize Maxwellian spacetime, which is supposed to be endowed with spatial and temporal metric structure and with a standard of rotation for smooth vector fields, but which is *not* supposed to pick out a preferred class of inertial trajectories—i.e., Maxwellian spacetime carries something less than a full affine structure. When Earman (1989) introduced Maxwellian spacetime, he defined it using an equivalence class of covariant derivative operators all agreeing on which smooth timelike vector fields are non-rotating[2]; Weatherall (2016c) adopted the same definition. But one might worry that this approach is problematic. In particular, defining Maxwellian spacetime by taking an equivalence class of derivative operators makes reference to structure that one does not attribute to spacetime. Manipulating a standard of rotation then involves choosing some derivative operator from the equivalence class, performing a calculation with that derivative operator, and then showing that the result of the calculation, if judiciously performed, is independent of the choice. But in such cases, one often encounters intermediate terms that *do* depend on the choice of derivative operator. How are we to interpret such terms—especially when they concern objects that represent physical magnitudes? The problem becomes particularly acute in light of Saunders' argument that Maxwellian spacetime may well be the correct setting for Newtonian gravitation. If so, one would like to be able to reason about quantities in Maxwellian spacetime without needing to introduce further structure.

A second (related) worry is implicit in recent remarks by Wallace (2016b), who points out that there is a more direct “Kleinian” characterization of the intended structure that, one might

E-mail address: weatherj@uci.edu.

[1] Of course, there is an older literature addressing the significance of Corollary VI—see, in particular, Stein (1967, 1977) and DiSalle (2008). There is also a literature on the closely related question of how to understand the relationship between “ordinary” Newtonian gravitation and “geometrized” Newtonian gravitation, also known as Newton-Cartan theory: see, for instance, Glymour (1980), Knox (2011), and Weatherall (2016a).

[2] Here and in what follows, we consider only torsion-free derivative operators.

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think, captures the intrinsic geometry more effectively than introducing an equivalence class of derivative operators.[3] More generally, Wallace argues that the example of Maxwellian spacetime, defined following Earman, shows that coordinate-free methods are not an intuitive way of characterizing certain spatiotemporal structures. But even setting aside the issue of what counts as more "intuitive", Wallace is surely correct that the definition of a "standard of rotation" used by Earman and others obscures the intrinsic geometry of Maxwellian spacetime. One might think, from experience with other examples, that it would preferable to have *both* Kleinian and coordinate-free characterizations of a standard of rotation that adequately capture this structure.

My purpose in this short note is to show that one can characterize a "standard of rotation" in just the sense that Earman and others discuss, in a fully covariant, coordinate-free manner, without ever introducing covariant derivative operators and with no equivalence classes in sight.[4] This structure permits an alternative characterization of Maxwellian spacetime that avoids the worries mentioned above. In particular I will go on to show how a standard of rotation can be used to explicitly define quantities, such as the rate of change of a spacelike vector field in a timelike direction, that are essential for performing standard manipulations in Maxwellian spacetime. Along the way, I make some remarks about spatial geometry in classical spacetime structures that may be of independent interest.[5]

In what follows, let M be a smooth four-manifold.[6] A *temporal metric* on M is a closed, non-vanishing one-form t_a; a *spatial metric* on M is a smooth, symmetric tensor field h^{ab}, which admits, at each point, a collection of four non-vanishing covectors $\overset{i}{\sigma}_a$, for $i = 0, 1, 2, 3$, such that $h^{ab}\overset{i}{\sigma}_a\overset{j}{\sigma}_b = 1$ if $i = j = 1, 2, 3$ and 0 otherwise. A temporal metric t_a and spatial metric h^{ab} are *compatible* if $h^{ab}t_b = 0$.[7] In what follows, we will limit attention to spatial metrics that are compatible with some temporal metric (or other). We will say that a covariant derivative operator ∇ on M is *compatible* with temporal and spatial metrics t_a and h^{ab} if $\nabla_a t_b = \mathbf{0}$ and $\nabla_a h^{bc} = \mathbf{0}$.

Fix a spatial metric h^{ab} on M. We will say that a vector ξ^a at a point p in M is *timelike* if there exists a non-vanishing covector τ_a such that $h^{ab}\tau_b = 0$ and $\xi^a\tau_a \neq 0$; otherwise it is *spacelike*.[8] It follows immediately that, at any point p, the spacelike vectors at p form a three dimensional subspace S_pM of the tangent space at p, T_pM. Given a temporal metric t_a, a timelike vector ξ^a will be called *unit* if $|\xi^a t_a| = 1$.

Let us now introduce the following notation.[9] Instead of using the usual Latin indices, we will write, for spacelike vectors and vector fields, underlined Latin indices, so that a spacelike vector ξ will be written $\xi^{\underline{a}}$. Likewise, given a linear functional λ acting on spacelike vectors, we will write $\lambda_{\underline{a}}$. Finally, we can consider tensor fields with (some) underlined indices, as in $\lambda^{a}{}_{\underline{c}}{}^{\underline{b}}{}_{d}$: in such cases, an underlined index appearing in a contravariant (raised) position indicates that, for any covector τ_a, if $h^{ab}\tau_b = 0$, then τ_a, contracted with that index, yields zero; meanwhile an underlined index appearing in a covariant (lowered) position indicates that the relevant action is restricted to spacelike vectors (i.e., it is not defined for timelike vectors). Note that we may always freely remove the lines under contravariant indices, since every spacelike vector at a point is in particular a vector at that point; and we may freely add lines under any covariant (lowered) indices, since every linear functional on tangent vectors at a point may be restricted to spacelike vectors at that point. Hence, we may write h^{ab} as $h^{\underline{ab}}$ and, for any temporal metric t_a, we have $t_{\underline{a}} = 0$. But we cannot generally add lines under contravariant indices, since not all tangent vectors are spacelike, and we cannot remove them from covariant indices, since linear functionals on spacelike vectors will not have unique extensions to all tangent vectors. We will call underlined indices *spatial indices*.

Given the structure defined so far, one can make sense of a *spatial derivative operator D* on M, which gives a standard for differentiation of smooth fields with (only) spatial indices in spacelike directions. I make this idea precise below, but the details are not essential for stating the main claim. The basic fact about spatial derivative operators that matters for what follows, proved in Prop. 2 below, is that given a spatial metric h^{ab}, there exists a unique spatial derivative operator D with the property that $D_{\underline{a}}h^{\underline{bc}} = 0$. Thus the structure already defined determines a unique spatial derivative operator, in much the same way that a pseudo-Riemannian metric determines a unique derivative operator.[10]

We can now make the central point. Fix a temporal metric t_a compatible with h^{ab}. A *standard of rotation* compatible with t_a and h^{ab} is a map $\circlearrowleft$ from smooth vector fields ξ^a on M to smooth, antisymmetric, rank $(2, 0)$ tensor fields $\circlearrowleft^n\xi^a$ on M, satisfying the following conditions:

1. $\circlearrowleft$ commutes with addition of smooth vector fields, i.e., given any two smooth vector fields ξ^a and η^a, $\circlearrowleft^n(\xi^a + \eta^a) = \circlearrowleft^n\xi^a + \circlearrowleft^n\eta^a$;
2. Given any smooth vector field ξ^a and any smooth scalar field α, $\circlearrowleft^n(\alpha\xi^a) = \alpha\circlearrowleft^n\xi^a + \xi^{[a}d^{n]}\alpha$[11];

[3] There is an issue, here, which is that alternative approaches all begin with a coordinate system, and then introduce a class of coordinate transformations that leave some structure invariant—a strategy that I understand as introducing extra structure—the coordinate system—and then removing it by taking equivalence classes. But I will not address this point in what follows.

[4] One might ask: could one do a similar thing in the case of a nondegenerate metric? (Or, put more baldly, why is this not a standard notion already?) The answer is "yes", but it is trivial, since every pseudo-Riemannian metric is compatible with a unique torsion-free derivative operator, and so one automatically gets *more* than a standard of rotation from the metric alone.

[5] Of course, this alternative formulation of Maxwellian spacetime only draws more attention to the question of whether this structure is sufficient to formulate Newtonian gravitational theory. One would like to find a coordinate-free presentation of the theory that makes use of precisely Maxwellian spacetime, as characterized here, and nothing else—a version, say, of Neil Dewar's "Maxwell gravitation" expressed using only a standard of rotation, (Dewar, 2017). I do not attempt that here, though see footnote 21 and the surrounding discussion for a first step in that direction.

[6] We assume all of the manifolds we consider are connected, paracompact, and Hausdorff.

[7] For a discussion of these notions, including an account of why the term "metric" is appropriate in each case, see Malament (2012, §4.1).

[8] Observe that we have defined our notion of timelike and spacelike in a way that does not refer to a temporal metric.

[9] This sort of "mixed index" notation is a generalization of the abstract index notation; it is described in more detail in, for instance, Weatherall (2016b); see also Geroch (1996).

[10] Note the difference from the presentation in Malament (2012, §4.1): he defines a spatial derivative operator, but does so only relative to (1) a specific temporal metric t_a and (2) a unit timelike vector field ξ^a; moreover, the spatial derivative operator he defines acts, in principle, on arbitrary smooth tensor fields on M. There is nothing wrong with this, of course, and I make use of the same construction in the Proof of Prop. 2. But it perhaps obscures the sense in which we get a unique spatial derivative operator from the spatial geometry alone, and given the purpose of the present note, it seems judicious to avoid any appearances of invoking structure beyond what is strictly needed.

[11] The operator d is the exterior derivative. Here and throughout, we raise indices on derivative operators with the spatial metric h^{ab}, i.e., $d^n\alpha = h^{mn}d_n\alpha$.