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Would two dimensions be world enough for spacetime?

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ABSTRACT

We consider various curious features of general relativity, and relativistic field theory, in two spacetime dimensions. In particular, we discuss: the vanishing of the Einstein tensor; the failure of an initial-value formulation for vacuum spacetimes; the status of singularity theorems; the non-existence of a Newtonian limit; the status of the cosmological constant; and the character of matter fields, including perfect fluids and electromagnetic fields. We conclude with a discussion of what constrains our understanding of physics in different dimensions.

1. Introduction

Philosophers of physics—and conceptually-oriented mathematical physicists—have gained considerable insight into the foundations and interpretation of our best physical theories, including general relativity, non-relativistic quantum theory, and quantum field theory, by studying the relationships between these theories and other "nearby" theories. For instance, one can better understand general relativity by studying its relationship to Newtonian gravitation, particularly in the form of geometrized Newtonian gravitation (i.e. Newton-Cartan theory)[1]; or by considering its relationship to other relativistic theories of gravitation.[2] Likewise, formulating classical mechanics in the language of Poisson manifolds provides important resources for understanding the structure of Hilbert space and quantum theory.[3] And thinking about classical field theory using nets of *-algebras on spacetime can help us better understand quantum field theory.[4]

The key feature of projects of the sort just described is that they are comparative: one draws out features of one theory by considering the ways in which it differs from other theories. But there is a closely allied project—or better, strategy for conceiving of projects—that, though often taken up by mathematical physicists, has received considerably less attention in the philosophy of physics literature.[5] This strategy is to study the foundations of a physical theory by considering features of that same theory in other dimensions. Doing so can provide insight into questions concerning, for instance, whether inferences about the structure of the world that make use of the theory in fact follow from the theory itself, or if they depend on ancillary assumptions. For instance, (vacuum) general relativity in four dimensions is, in a certain precise sense, deterministic. But as we argue in what follows, this feature depends on dimensionality; in two dimensions the theory, at least on one understanding, does not have a well-posed initial value formulation.

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[1] For background on geometrized Newtonian gravitation, see Trautman (1965) and (especially) Malament (2012, Ch. 4). For projects that aim to use this theory to provide new insight into general relativty, see, for instance, Cartan (1923, 1924), Friedrichs (1927), Friedman (1983), Weatherall (2011, 2014, 2017a, 2017b), Weatherall and Manchak (2014), Dewar and Weatherall (2017), and Ehlers (1997).

[2] See, for instance, Brown (2005), Knox (2011, 2013), Pitts (2016), or Weatherall (2017a).

[3] See, for instance, Weyl (1950) and Landsman (1998, 2017) for mathematical treatments of the main issues; for examples of how these ideas have been applied by philosophers, see, for instance, Feintzeig (2016a) and Feintzeig, (Le)Manchak, Rosenstock, and Weatherall (2017).

[4] See, for instance, Brunetti, Fredenhagen, and Ribeiro (2012), Rejzner (2016), and Feintzeig (2016b,c).

[5] To our knowledge, the projects that come closest to this strategy are those that evaluate arguments that spacetime must have a certain dimensionality (Callender, 2005); or those that consider the details of constructive quantum field theory, which often considers lower-dimensional models (Hancox-Li, 2017; Ruetsche, 2011).

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A detailed study of the physics of different dimensions can also reveal striking disanalogies between physics in different dimensions, which can then inform other projects. For instance, it is common in the mathematical physics literature to consider quantizing field theories—including general relativity—in lower dimensions.[6] Doing so can provide important hints at what a full theory of quantum gravity, say, might look like. Moreover, there is a temptation to try to draw preliminary philosophical morals about our own universe from these quantum theories in lower dimensions—particularly among philosophers who prefer to work with mathematically rigorous formulations of theories, which in the case of quantum field theories are only available in lower dimensions. But there are also reasons to be cautious about such hints: if classical theories, including general relativity, have very different features in different dimensions, the inferences we can draw about their quantum counterparts in those other dimensions may not carry over to the four dimensional case.

In what follows, we investigate the features of general relativity in two spacetime dimensions, on several ways of understanding what that might mean. In the first instance, we suppose that Einstein's equation holds in all dimensions. As we will show, the resulting theory is strikingly different, in a number of important ways, from the standard four dimensional theory. Of course, that theories can differ dramatically in different dimensions is hardly news—especially to the experts in mathematical physics who work on these theories in fewer (or more) than four dimensions—and it is well-known that general relativity in two dimensions is “pathological” or (arguably) “trivial”. But there are some features that we discuss below that, to our knowledge, have not been drawn out in detail in the literature—including, for instance, the status of the initial value formulation and the non-existence of a Newtonian limit (where Newtonian gravitation is generalized by assuming that the geometrized Poisson equation holds in all dimensions). Moreover, in our view it is valuable to collect these features of the two-dimensional theory together in one place, and to reflect on what they can teach us about the structure of general relativity more generally. They also raise the question of what it means to identify theories across dimensions, particularly when the ostensibly “same” theory can have very different qualitative features in different dimensions.

In the next section, we will discuss the status of the Einstein tensor—which vanishes identically in two dimensions—and Einstein's equation (without cosmological constant). In a sense, this is the principal feature of two-dimensional general relativity from which the other strange features follow. In the following section, we will discuss the status of the initial value formulation and singularity theorems in two dimensions. Next we will consider Newtonian gravitation in two dimensions, generalized as noted above, and show that it is not the classical limit of general relativity. In the following section, we will consider what happens when one includes a cosmological constant, exploring the consequences for the character of some matter fields in two dimensions. We will then discuss what it means to generalize a theory to different dimensions, by considering various arguments about alternative formulations of the theory in two dimensions. We conclude by arguing that the discussion here of how to generalize a theory to other dimensions raises questions for a common view according to which to interpret a physical theory is to characterize the space of possibilities allowed by that theory.

2. Einstein's tensor and Einstein's equation

We begin with a few preliminaries concerning the relevant background formalism of general relativity.[7] An n-dimensional relativistic *spacetime* (for $n \geq 2$) is a pair (M, g_{ab}) where M is a smooth, connected n-dimensional manifold and g_{ab} is a smooth, non-degenerate, pseudo-Riemannian metric of Lorentz signature $(+, -, \ldots, -)$ defined on M.[8]

For each point $p \in M$, the metric assigns a cone structure to the tangent space M_p. Any tangent vector ξ^a in M_p will be *timelike* if $g_{ab}\xi^a\xi^b > 0$, *null* if $g_{ab}\xi^a\xi^b = 0$, or *spacelike* if $g_{ab}\xi^a\xi^b < 0$. Null vectors delineate the cone structure; timelike vectors are inside the cone while spacelike vectors are outside. A *time orientable* spacetime is one that has a continuous timelike vector field on M. A time orientable spacetime allows one to distinguish between the future and past lobes of the light cone. In what follows, it is assumed that spacetimes are time orientable and that an orientation has been chosen.

For some open (connected) interval $I \subseteq \mathbb{R}$, a smooth curve $\gamma : I \to M$ is *timelike* if the tangent vector ξ^a at each point in $\gamma[I]$ is timelike. Similarly, a curve is *null* (respectively, *spacelike*) if its tangent vector at each point is null (respectively, spacelike). A curve is *causal* if its tangent vector at each point is either null or timelike. A causal curve is *future directed* if its tangent vector at each point falls in or on the future lobe of the light cone. A curve $\gamma : I \to M$ in a spacetime (M, g_{ab}) is a *geodesic* if $\xi^a \nabla_a \xi^b = 0$, where ξ^a is the tangent vector to γ and ∇_a is the unique derivative operator compatible with g_{ab}.

The fundamental dynamical principle of general relativity is known as *Einstein's equation*. In four dimensions, Einstein's equation may be written, without cosmological constant, in natural units as

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}. \tag{2.1}$$

Here $R_{ab} = R^n{}_{abn}$ is the Ricci tensor associated with g_{ab} and $R = R^a{}_a$ is the curvature scalar. The left-hand side of this equation is known as the *Einstein tensor*, often written G_{ab}; the right-hand side is the sum of the energy-momentum tensors associated with all matter present in the universe and their interactions.

In the first instance, we generalize general relativity to other dimensions by taking this expression to relate curvature and energy-momentum in arbitrary dimensions (We will return to this proposal in sections 5 and 6 and consider other possibilities.). In particular, define, in a spacetime of any dimension, the Einstein tensor to be $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$.

We have the following immediate proposition.

Proposition 1. *Let (M, g_{ab}) be a two-dimensional spacetime. Then $R_{ab} = \frac{1}{2} R g_{ab}$ and $G_{ab} = 0$.*

Proof. Given a pseudo-Riemannian manifold of any dimension $n \geq 2$, the Riemann tensor $R_{abcd} = g_{an} R^n{}_{bcd}$ is antisymmetric in the first two indices and in the last two indices: $R_{abcd} = R_{[ab][cd]}$. It follows that R_{abcd} can be written as a linear combination of outer products of two-forms. But the space of two-forms on a two-dimensional manifold is one-dimensional, and so we have $R_{abcd} = f \varepsilon_{ab} \varepsilon_{cd}$, where ε_{ab} is either volume element on M

[6] See, for instance, Glimm and Jaffe (1987); for a discussion of quantum gravity in particular, see Carlip (2003).

[7] The reader is encouraged to consult Hawking and Ellis (1973), Wald (1984), and Malament (2012) for details.

[8] We also assume M to be Hausdorff and paracompact. All objects that are candidates to be smooth in what follows are assumed to be so, even when not mentioned explicitly.

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