Journal of Geometry and Physics 134 (2018) 48–57

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Vector-valued (super) weaving frames

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A R T I C L E   I N F O

Article history:
Received 20 April 2018
Received in revised form 23 July 2018
Accepted 24 July 2018
Available online xxxx

MSC:
primary 42C15
secondary 42C30
42C40

Keywords:
Frames
Vector-valued frames
Weaving frames
Riesz basis

A B S T R A C T

Two frames $\{\phi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ for a separable Hilbert space H are woven if there are positive constants $A \leq B$ such that for every subset $\sigma \subset I$, the family $\{\phi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for H with frame bounds A, B. Bemrose et al. introduced weaving frames in separable Hilbert spaces and observed that weaving frames have potential applications in signal processing. Motivated by this, and the recent work of Balan in the direction of application of vector-valued frames (or superframes) in signal processing, we study vector-valued weaving frames. In this paper, first we give some fundamental properties of vector-valued weaving frames. It is shown that if a family of vector-valued frames is woven, then the corresponding family of frames for atomic spaces is woven, but the converse is not true. We present a technique for the construction of vector-valued woven frames from given woven frames for atomic spaces . Necessary and sufficient conditions for vector-valued weaving Riesz sequences are given. Several numerical examples are given to illustrate the results.

1. Introduction

Duffin and Schaeffer [1] introduced frames for Hilbert spaces in the context of nonharmonic Fourier series. Today frames have applications in a wide range of areas in applied mathematics. The applications of frames in signal processing are now well-known, for example see [2,3]. Balan [4] introduced the concept of a vector-valued frame (or "superframe") in the context of multiplexing of signals and further studied in [5]. The vector-valued frame has significant applications in mobile communication, satellite communication, and computer area network. Recently, Bemrose, Casazza, Gröchenig, Lammers and Lynch in [6] proposed weaving frames in a separable Hilbert space. The concept of weaving frames is motivated by a problem regarding distributed signal processing where redundant building blocks (frames) play an important role. For example, in wireless sensor networks where frames may be subjected to distributed processing under different frames. Motivated by the concept of weaving frames and superframes and their application in Gabor and wavelet analysis, in this paper, we study vector-valued (super) weaving frames. Weaving frames have potential applications in wireless sensor networks that require distributed processing under different frames, as well as preprocessing of signals using Gabor frames. Recently, M. de Gosson [7] studied "deformation of Gabor frames" which has potential applications in physics; and related with the notion of weaving frames in the context of deformation of a lattice. Notable contribution in the paper is a new technique for the construction of vector-valued weaving frames from frames of atomic spaces. Some necessary and sufficient conditions for vector-valued weaving Riesz sequences are given. Finally, a result for vector-valued weaving Riesz sequences in terms of operators on atomic spaces has been obtained.

1.1. Previous works on weaving frames

For a positive integer m, we write $[m] = \{1, 2, \ldots, m\}$. We start with the definition of weaving frames in separable Hilbert spaces.

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https://doi.org/10.1016/j.geomphys.2018.07.021

Definition 1.1 (*[6]*)**.** A family of frames $\{\phi_{ij}\}_{i\in I}$ for $j \in [m]$ for a Hilbert space H is said to be *woven* if there are universal constants A and B, so that for every partition $\{\sigma_j\}_{j\in[m]}$ of I, the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for H with lower and upper frame bounds A and B, respectively.

Definition 1.2 (*[6]*)**.** A family of frames $\{\phi_{ij}\}_{i\in\mathbb{N}, j\in[m]}$ for a Hilbert space H is said to be *weakly woven* if for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{N}$, the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for H.

Remark 1.3. It is proved in [6] that this weaker form of weaving (given in Definition 1.2) is equivalent to weaving

Bemrose et al. proved in [6] that a frame (which is not a Riesz basis) cannot be weave with a Riesz basis.

Theorem 1.4 ([6])**.** *Let $\{\phi_j\}_{j=1}^{\infty}$ be a Riesz basis and let $\{\psi_j\}_{j=1}^{\infty}$ be a frame for H. If $\{\phi_j\}_{j=1}^{\infty}$ and $\{\psi_j\}_{j=1}^{\infty}$ are woven, then $\{\psi_j\}_{j=1}^{\infty}$ must actually be a Riesz basis.*

Bemrose et al. [6] classified when Riesz bases and Riesz basic sequences can be woven and provide a characterization in terms of distances between subspaces. Furthermore, they proved that if two Riesz bases are woven, then every weaving is in fact a Riesz basis, and not just a frame. A geometric characterization of woven Riesz bases in terms of distance between subspaces of a Hilbert space H is given in [6]. Casazza and Lynch in [8] reviewed fundamental properties of weaving frames. They considered a relation of frames to projections and gave a better understanding of what it really means for two frames to be woven. Finally, they discussed a weaving equivalent of an unconditional basis. Deepshikha and Vashisht [9] studied infinitely woven frames in separable Hilbert spaces. They showed that there exists an infinite family of frames for which finite weaving is possible, but the family itself is not a frame under infinite weaving. Some results related to infinitely woven Riesz bases can be found in [9]. Generalized weaving frames in separable Hilbert spaces were studied in [10–12]. Some necessary and sufficient conditions for weaving fusion frames may found in [13]. In [14,15], Vashisht and Deepshikha studied weaving frames with respect to measure spaces. Dörfler and Faulhuber [16] studied weaving Gabor frames in $L^2(\mathbb{R})$. They also discussed a family of localization operators related to weaving Gabor frames. A sufficient criteria for a family of multi-window Gabor frames to be woven may found in [16]. Casazza, Freeman and Lynch [17] extended the concept of weaving Hilbert space frames to the Banach space setting. They introduced and studied *weaving Schauder frames* in Banach spaces.

1.2. Overview and main results

The paper is organized as follows: Section 2 contains the basic definitions and results about frames, weaving frames and vector-valued frames. In Section 3, we introduce vector-valued weaving frames. Some properties of vector-valued frames are given. It is shown that frames associated with atomic spaces are woven, provided the given family of vector-valued frames is woven, see Proposition 3.5. The converse is not true. Theorem 3.9 gives a technique for the construction of vector-valued woven frames from woven frames for atomic spaces. In Section 4, we give necessary and sufficient conditions for vector-valued weaving Riesz sequences. Proposition 4.4 gives a sufficient condition for vector-valued weaving Riesz bases. It is shown in Theorem 4.7 that if a family of frames associated with one of the atomic spaces is a woven Riesz sequence, then the family of vector-valued Bessel sequences is also a woven Riesz sequence. Finally, sufficient conditions for vector-valued weaving Riesz sequences in terms of bounded operators on atomic spaces are given in Theorem 4.9. Several examples and counter-example are given to illustrate our results.

2. Preliminaries

In this section, we review the concepts of frames, vector-valued frames and weaving frames. We begin with some notations: The set of all positive integers is denoted by $\mathbb{N}$ or $\mathbb{Z}^+$. By $\mathbb{Z}$ and $\mathbb{Z}^-$ we denote the set of all integers and negative integers, respectively and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. For a set E, χ_E denote the characteristic function of E. By $\mathbb{K}$ we denote field of real or complex numbers. For an indexing set I, as usual $\ell^2(I)$ is the space of all square summable complex-valued sequences indexed by the set I. The space of bounded linear operators from a normed space X into a normed space Y is denoted by $\mathcal{B}(X, Y)$. If $X = Y$, then we write $\mathcal{B}(X, Y) = \mathcal{B}(X)$.

2.1. Hilbert space frames

A countable sequence $\{f_k\}_{k\in I}$ in a separable Hilbert space H is called a *frame* (or *Hilbert frame*) for H if there exist positive numbers $\alpha_o \le \beta_o < \infty$ such that

$$\alpha_o\|f\|^2 \le \sum_{k\in I} |\langle f, f_k\rangle|^2 \le \beta_o\|f\|^2 \text{ for all } f \in H. \tag{2.1}$$

The numbers α_o and β_o are called *lower* and *upper frame bounds*, respectively. If upper inequality in (2.1) is satisfied, then we say that $\{f_k\}_{k\in I}$ is a *Bessel sequence* (or *Hilbert Bessel sequence*) with *Bessel bound* β_o. The frame $\{f_k\}_{k\in I}$ is *tight* if it is possible to choose $\alpha_o = \beta_o$.