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Scalar products of the elliptic Felderhof model and elliptic Cauchy formula

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A B S T R A C T

We analyze the scalar products of the elliptic Felderhof model introduced by Foda–Wheeler–Zuparic as an elliptic extension of the trigonometric face-type Felderhof model by Deguchi–Akutsu. We derive the determinant formula for the scalar products by applying the Izergin–Korepin technique developed by Wheeler to investigate the scalar products of integrable lattice models. By combining the determinant formula for the scalar products with the recently-developed Izergin–Korepin technique to analyze the wavefunctions, we derive a Cauchy formula for elliptic Schur functions.

1. Introduction

Elliptic integrable models are classes of integrable models described by elliptic functions. Investigations of elliptic integrable models lead to new discoveries of mathematical structures. An instance is the notion of elliptic quantum groups [1–4] which are extensions of the quantum groups [5–7], introduced through the analysis of the eight-vertex model, eight-vertex solid-on-solid model and their generalizations [8–12]. Recently, there are also progresses on partition functions of the eight-vertex solid-on-solid models [13–26] from the viewpoint of the quantum inverse scattering method [27,28], vertex operator method [29] and so on.

In this paper, we investigate another class of elliptic integrable model. We analyze the scalar products of the elliptic Felderhof model introduced by Foda–Wheeler–Zuparic [30] as an extension of the face-type Felderhof model [31] by Deguchi–Akutsu [32]. The elliptic Felderhof model (Foda–Wheeler–Zuparic model), and its closely related elliptic Perk–Schultz model (Okado–Deguchi–Fujii–Martin model) constructed by Okado [33], Deguchi–Fujii [34] and Deguchi–Martin [35] as an elliptic extension of the Perk–Schultz model [36], are interesting models to be investigated, since the corresponding trigonometric models were discovered by number theorists recently to be related with automorphic representation theory and deformations of Weyl character formulas (Tokuyama formulas) for symmetric functions. Bump–Brubaker–Friedberg [37] constructed free-fermion models by themselves and showed that the wavefunctions are given

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as a product of a deformed Vandermonde determinant and Schur functions. One of the consequences of their results is the natural construction of the Tokuyama formula [38] as wavefunctions of integrable models, which is a one-parameter deformation of the Weyl character formula for the Schur functions. Their result is the one of the main motivations to study the elliptic Felderhof model of Foda–Wheeler–Zuparic and the elliptic Perk–Schultz model of Okado, Deguchi–Fujii and Deguchi–Martin, since these models can be regarded as elliptic analogues of the free-fermion model which Bump–Brubaker–Friedberg introduced and analyzed (the quantum group structure of the trigonometric models can be found in [32,39,40] for example). There are not so much studies on the partition functions of these elliptic models. Foda–Wheeler–Zuparic showed the factorization of the domain wall boundary partition functions of these models [30] by applying the Izergin–Korepin technique [41,42], which is a classical method to analyze the domain wall boundary partition functions of integrable models. Recently, we extended the Izergin–Korepin technique to be able to analyze the wavefunctions [43–45], and showed that the wavefunctions of these elliptic models are given as a deformed elliptic Vandermonde determinant and elliptic symmetric functions which can be viewed as elliptic Schur functions (see Schlosser [46] or Noumi [47,48] for other types of elliptic Schur functions introduced from the viewpoint of combinatorics, special functions and classical integrable systems). The results can be viewed as elliptic analogues of the one by Bump–Brubaker–Friedberg.

In this paper, we investigate another special class of partition functions called the scalar products. One of the motivations to study this class of partition functions comes from the recent active line of researches on the application of the correspondence between symmetric functions and wavefunctions of integrable models to derivations of various algebraic identities. For the free-fermionic models, see [49–64] for examples on integrability approach to symmetric functions, as well as closely related non-intersecting lattice paths approach. There are also investigations on the six-vertex models and face models related to the XXX, XXZ and XYZ quantum integrable spin chains and q-boson models, where the Schur, Grothendieck, Hall-Littlewood polynomials and their generalizations appear as the wavefunctions. See [65–78] for examples on various studies of these models. Among these active studies, it was realized that the analysis on the scalar products lead us to Cauchy formulas for symmetric functions. Directly evaluating the scalar products to get determinant formulas in one way, and comparing the expressions with another way of evaluation by inserting completeness relation and express it as the sum of products of the wavefunctions whose explicit forms are given by symmetric functions, one can get Cauchy formulas for symmetric functions. This quantum integrability approach often enables us to derive algebraic identities which are almost impossible to find by any other means. In this paper, we apply this idea to the elliptic Felderhof model, and derive the Cauchy formula for the elliptic Schur functions. The main part of this paper is the direct evaluation of the scalar products. We apply the Izergin–Korepin technique developed by Wheeler [79] to derive the scalar products of integrable models. In his paper, Wheeler showed that his technique can be applied to the $U_q(sl_2)$ six-vertex model to derive the Slavnov's determinant formula [80] for the XXZ spin chain for example, by introducing and listing the properties which uniquely defines the intermediate scalar products, and showing the explicit determinant forms satisfying all the properties. We apply his technique to the elliptic Felderhof model and obtain the determinant formula for the scalar products. Together with our results on the correspondence between the wavefunctions and the elliptic Schur functions [44,45] obtained by the Izergin–Korepin analysis on the wavefunctions, we derive the Cauchy formula for the elliptic Schur functions.

This paper is organized as follows. In Section 2, we recall the properties of theta functions and the Foda–Wheeler–Zuparic (elliptic Felderhof) model. In Section 3, we introduce the scalar products, and derive the determinant formula by applying the Izergin–Korepin technique developed by Wheeler. In Section 4, by combining with another evaluation of the scalar products using the correspondence between the wavefunctions and the elliptic Schur functions, we derive the Cauchy formula for the elliptic Schur functions. Section 5 is devoted to the conclusion of this paper.

2. Foda–Wheeler–Zuparic (elliptic Felderhof) model

In this section, we first introduce elliptic functions and list their properties, and introduce the Foda–Wheeler–Zuparic model which is an elliptic analogue of the Felderhof model. The theta functions $H(u)$ is

$$H(u)=2\sinh u\prod_{n=1}^{\infty}(1-2\mathbf{q}^{2n}\cosh(2u)+\mathbf{q}^{4n})(1-\mathbf{q}^{2n}), \tag{2.1}$$

where $\mathbf{q}$ is the elliptic nome ($0<\mathbf{q}<1$). For the description of the matrix elements of the dynamical R-matrix of the elliptic Felderhof model, we introduce the following notation

$$[u]=H(\pi iu). \tag{2.2}$$

The theta function $[u]$ is an odd function $[-u]=-[u]$ and satisfies the quasi-periodicities

$$[u+1]=-[u], \tag{2.3}$$

$$[u-i\log(\mathbf{q})/\pi]=-\mathbf{q}^{-1}\exp(-2\pi iu)[u]. \tag{2.4}$$

We use the following property about the elliptic polynomials [13,81] presented below.

A character is a group homomorphism χ from multiplicative groups $\Gamma=\mathbf{Z}+\tau\mathbf{Z}$ to $\mathbf{C}^\times$. An N-dimensional space $\Theta_N(\chi)$ is defined for each character χ and positive integer N, which consists of holomorphic functions $\phi(y)$ on $\mathbf{C}$ satisfying the