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Essentially normal composition operators on H^2

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ABSTRACT

We prove a simple criterion for essential normality of composition operators on the Hardy space induced by maps in a reasonably large class $\mathcal{S}$ of analytic self-maps of the unit disk. By combining this criterion with boundary Carathéodory–Fejér interpolation theory, we exhibit a parametrization for all rational self-maps of the unit disk which induce essentially normal composition operators.

1. Introduction

For φ an analytic self-map of the unit disk $\mathbb{D}$, the composition operator $C_\varphi\colon f \to f \circ \varphi$ induced by φ is a bounded operator on the Hardy space H^2. A bounded operator A is said to be essentially normal if its self-commutator $[A^*, A] = A^*A - AA^*$ is a compact operator, and trivially essentially normal if A is either normal ($[A^*, A] = 0$) or compact. Normal composition operators on H^2 were characterized by Schwartz [26] and compact composition operators by Shapiro [27] and, via a different criterion, by Sarason [24] and Shapiro–Sundberg [28]; see also Cima–Matheson [10]. In [7], Bourdon–Levi–Narayan–Shapiro characterize the class of linear fractional self-maps of $\mathbb{D}$ that induce non-trivially essentially normal composition operators; these maps are exactly the parabolic non-

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automorphisms of $\mathbb{D}$. These authors provide additional examples of essentially normal composition operators induced by maps which, like linear fractional non-automorphisms of $\mathbb{D}$, have order of contact 2 with $\partial\mathbb{D}$ at one point. To the best of the author's knowledge, no non-trivially essentially normal composition operators with inducing maps having order of contact $n > 2$ with $\partial\mathbb{D}$ were known prior to the present work.

In this paper we prove a simple criterion (Theorem 6.7 below) for essential normality of composition operators induced by maps φ in the class $\mathcal{S}$ introduced by Kriete–Moorhouse in [21]. Roughly speaking, the class $\mathcal{S}$ consists of analytic self-maps φ of $\mathbb{D}$ that have "significant contact" with $\partial\mathbb{D}$ at only a finite number of points, with φ having "sufficient derivative data" at every such point. As a corollary, we show that for a self-map φ of $\mathbb{D}$ which extends analytically to a neighborhood of $\overline{\mathbb{D}}$, C_φ is non-trivially essentially normal if and only if φ fixes one point of $\partial\mathbb{D}$, has derivative equal to 1 there, and maps the rest of $\partial\mathbb{D}$ into $\mathbb{D}$. Note that for the linear fractional case this criterion is equivalent to the characterization described above. Our criterion, in conjunction with boundary Carathéodory–Fejér interpolation theory, yields a parametrization for all rational self-maps φ of $\mathbb{D}$ that induce non-trivially essentially normal C_φ on H^2.

We rely on results from three distinct areas, presented in Sections 3–5. First, we explore a special case of a boundary version of the Carathéodory–Fejér problem studied by Agler–Lykova–Young in [1,2]. Second, we discuss relations in the Calkin Algebra using results by Kriete–Moorhouse [21]. In particular, we derive a decomposition of a composition operator modulo the ideal $\mathcal{K}$ of compact operators into a sum of composition operators induced by "basic" rational functions (Theorem 4.4). Third, using formulas and ideas from Bourdon–Shapiro [8], based on work of Cowen–Gallardo [14] and Hammond–Moorhouse–Robbins [17], we obtain an operator formula for $C_\psi C_\varphi^*$ where φ is rational and ψ is an auxiliary map, and reduce this formula modulo $\mathcal{K}$. Additionally, Faà di Bruno's formula, an identity generalizing the chain rule, plays a significant role.

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2. Preliminaries

2.1. Generalized chain rule – Faà di Bruno's formula

Faà di Bruno's formula is an identity generalizing the chain rule that has been known since 1800. The following is the statement of the formula in combinatorial form.

Theorem 2.1 *(Faà di Bruno's formula). [19] If g is analytic at z and f is analytic at $g(z)$, then*

$$(f \circ g)^{(k)}(z) = \sum_{\pi \in \Pi} f^{(|\pi|)}(g(z)) \cdot \prod_{B \in \pi} g^{(|B|)}(z), \tag{2.1}$$

where Π is the set of partitions of $\{1, \ldots, k\}$.