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Infinite-dimensional Bayesian approach for inverse scattering problems of a fractional Helmholtz equation

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A B S T R A C T

This paper focuses on a fractional Helmholtz equation describing wave propagation in the attenuating medium. According to physical interpretations, the fractional Helmholtz equation can be divided into loss- and dispersion-dominated fractional Helmholtz equations. In the first part of this work, we establish the well-posedness of the loss-dominated fractional Helmholtz equation (an integer- and fractional-order mixed elliptic equation) for a general wavenumber and prove the Lipschitz continuity of the scattering field with respect to the scatterer. Meanwhile, we only prove the well-posedness of the dispersion-dominated fractional Helmholtz equation (a high-order fractional elliptic equation) for a sufficiently small wavenumber due to its complexity. In the second part, we generalize infinite-dimensional Bayesian inverse theory to allow a part of the noise depends on the target function (the function that needs to be estimated). We also prove that the estimated function tends to be the true function if both the model reduction error and the white noise

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vanish. We eventually apply our theory to the loss-dominated model with an absorbing boundary condition.

1. Introduction

Numerous physical models have been proposed [1,44,36] to describe the attenuation effect which is an important phenomenon for considering wave propagation in some attenuating medium. When studying scattering problems with attenuating medium, researchers usually focus on the following Helmholtz equation:

$$\Delta u + k^2 n(x) u = 0, \tag{1.1}$$

where u denotes the wavefield, k denotes the wavenumber, and n denotes the refractive index with an imaginary component [13]. In other words, the analysis of scattering problems with absorbing medium can be incorporated into the classical studies on Helmholtz equations [2,4]. However, the attenuation effect actually incorporates two effects, namely, amplitude loss and velocity dispersion. The aforementioned model (1.1) mixes these two effects together. Hence, the attenuation effect can hardly be compensated when we handle some inverse problems, such as reverse-time migration [47].

We consider the space fractional wave equations proposed in [50] that can separate the two effects incorporated in the attenuation effect. Before revealing the form of this new fractional model, we introduce the time fractional wave equation. Based on Caputo's fractional derivative [40], the isotropic stress-strain ($\sigma - \epsilon$) relation can be deduced in the following form [10]:

$$\sigma = \frac{M_0}{t_0^{-2\gamma}} \frac{\partial^{2\gamma} \epsilon}{\partial t^{2\gamma}},$$

where M_0 is the bulk modulus and t_0 is a reference time. The following wave equation with Caputo's fractional derivative can then be established:

$$\frac{\partial^{2-2\gamma(x)}}{\partial t^{2-2\gamma(x)}} u = c(x)^2 \omega^{-2\gamma(x)} \Delta u, \tag{1.2}$$

where $c^2(x) = c_0^2(x) \cos^2(\pi\gamma(x)/2)$ and c_0 is the sound velocity. Denote $Q(x)$ to be the quality factor that is commonly used to characterize seismic attenuation. In this case, the fractional-order $\gamma(x)$ relates to the quality factor as follows:

$$\gamma(x) = \frac{1}{\pi} \arctan\left(\frac{1}{Q(x)}\right). \tag{1.3}$$