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Compactness of solutions to nonlocal elliptic equations

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ABSTRACT

We show that all nonnegative solutions of the critical semilinear elliptic equation involving the regional fractional Laplacian are locally universally bounded. This strongly contrasts with the standard fractional Laplacian case. Secondly, we consider the fractional critical elliptic equations with nonnegative potentials. We prove compactness of solutions provided the potentials only have non-degenerate zeros. Corresponding to Schoen's Weyl tensor vanishing conjecture for the Yamabe equation on manifolds, we establish a Laplacian vanishing rate of the potentials at blow-up points of solutions.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. The regional fractional Laplace operator is defined as

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$$(-\Delta_\Omega)^\sigma u(x) := \mathrm{P.V.}c_{n,\sigma} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}\,\mathrm{d}y \quad \text{for } u \in C^2(\Omega),$$

where $0 < \sigma < 1$ is a parameter, $c_{n,\sigma} = \frac{2^{2\sigma}\sigma\Gamma(\frac{n+2\sigma}{2})}{\pi^{\frac{n}{2}}\Gamma(1-\sigma)}$. The regional fractional Laplacian arises, for instance, from the Feller generator of the reflected symmetric stable process, see Bogdan–Burdzy–Chen [3], Chen–Kumagai [12], Guan–Ma [24], Guan [23], Mou–Yi [38] and many others. Here we are interested in universal boundness of positive solutions to nonlinear Poisson equation involving the regional fractional Laplacian. Making use of the standard blow-up argument of Gidas–Spruck [19] and the Liouville theorem, one can show that all nonnegative solutions of the equation $(-\Delta_\Omega)^\sigma u(x) = u^p$ with $1 < p < \frac{n+2\sigma}{n-2\sigma}$ are locally universally bounded. In view of the fractional Sobolev inequality, for p in that range we say the equation is subcritical. In contrast, the critical equation ($p = \frac{n+2\sigma}{n-2\sigma}$) has blow-up solutions when $\Omega = \mathbb{R}^n$. See Jin–Li–Xiong [27,28] and references therein for more discussions.

However, if Ω has nontrivial complement, we have

Theorem 1.1. *Suppose that Ω is an open subset of $\mathbb{R}^n$ and the measure of $\mathbb{R}^n \setminus \Omega$ is non-zero. Without loss of generality, suppose that the unit ball $B_1 \subset \Omega$. Let $u \in C^2(\Omega)$ be a nonnegative solution of*

$$(-\Delta_\Omega)^\sigma u = u^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{in } B_1. \tag{1}$$

If $n \geq 4\sigma$, then

$$\|u\|_{C^2(B_{1/2})} \leq C(n,\sigma,\Omega),$$

where $C(n,\sigma,\Omega) > 0$ is a constant depending only on n,σ,Ω.

Theorem 1.1 is of nonlocal nature and fails when $\sigma = 1$. Since no condition is assumed on solutions in the complement of B_1, there exist infinitely many solutions of (1). Note that (1) is the Euler–Lagrange equation of the fractional Sobolev inequality in Ω. Recently, Frank–Jin–Xiong [18] showed that the best constants of fractional Sobolev inequality depend on domains and can be achieved in many cases, which is different from the classical Sobolev inequalities in domains.

For every smooth bounded function u defined in Ω, by extending u to zero outside Ω we see that

$$(-\Delta_\Omega)^\sigma u(x) = (-\Delta)^\sigma u(x) - A_\Omega(x)u(x) \quad \text{for } x \in \Omega, \tag{2}$$

where $(-\Delta)^\sigma := (-\Delta_{\mathbb{R}^n})^\sigma$ is the standard fractional Laplacian,

$$A_\Omega(x) := c_{n,\sigma} \int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+2\sigma}}\,\mathrm{d}y. \tag{3}$$

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