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Quantitative uniqueness of solutions to parabolic equations

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A B S T R A C T

We investigate the quantitative uniqueness of solutions to parabolic equations with lower order terms on compact smooth Riemannian manifolds. Quantitative uniqueness is a quantitative form of strong unique continuation property. We characterize quantitative uniqueness by the rate of vanishing. We can obtain the sharp vanishing order for solutions in term of the $C^{1,1}$ norm of the potential functions, as well as the L^∞ norm of the coefficient functions. Some new quantitative Carleman estimates and three cylinder inequalities are established.

1. Introduction

In this paper, we study the quantitative uniqueness for parabolic equations with non-trivial lower order terms on compact smooth manifolds. Suppose u is a non-trivial solution to

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$$\triangle_g u - \partial_t u - \tilde{V}(x,t)u = 0 \quad \text{on } \mathcal{M}^1, \tag{1.1}$$

where $\mathcal{M}^1 = \mathcal{M} \times (-1, 1)$ and g is the metric on the compact smooth Riemannian manifold $\mathcal{M}$ with dimension $n \geq 2$. Assume that $\tilde{V} \in C^{1,1}$, where $\|\tilde{V}\|_{C^{1,1}(\mathcal{M}^1)} = \sup_{\mathcal{M}^1} |\tilde{V}| + \sup_{\mathcal{M}^1} |\nabla \tilde{V}| + \sup_{\mathcal{M}^1} |\partial_t \tilde{V}|$. We may also assume that $\|\tilde{V}\|_{C^{1,1}(\mathcal{M}^1)} \leq M$ for $M \geq 1$. Quantitative uniqueness, also called as quantitative unique continuation, is described by the vanishing order and used to characterize how much the solution vanishes. It is a quantitative way to describe the strong unique continuation property. If the condition that solution vanishes of infinite order at a point implies that the solution vanishes identically, then we say the strong unique continuation property holds.

Let's first review the progresses about quantitative uniqueness for elliptic equations. Recently, there are much attention in this topic. The most interesting example for quantitative unique continuation arises from the study of nodal sets for eigenfunctions on manifolds. For classical eigenfunctions on a compact smooth Riemannian manifold $\mathcal{M}$,

$$-\triangle_g \phi_\lambda = \lambda \phi_\lambda \qquad \text{in } \mathcal{M}, \tag{1.2}$$

Donnelly and Fefferman in [8] obtained that the maximal vanishing order of ϕ_λ is everywhere less than $C\sqrt{\lambda}$, here C only depends on the manifold $\mathcal{M}$. Such vanishing order for eigenfunction ϕ_λ is sharp, which can be verified from spherical harmonics.

Kukavica in [19] studied the quantitative unique continuation for Schrödinger equation

$$-\triangle u + V(x)u = 0. \tag{1.3}$$

If $\|V\|_{C^1} \leq K$ for some large constant $K > 1$, Kukavica showed that the upper bound of vanishing order is less than CK. From Donnelly and Fefferman's work in the case $V(x) = -\lambda$, this upper bound is not optimal. Recently, by different methods, the sharp vanishing order for solutions of (1.3) is shown to be less than $CK^{\frac{1}{2}}$ independently by Bakri in [2] and Zhu in [28]. The $\frac{1}{2}$ power of K matches the optimal result for the vanishing order of eigenfunctions in Donnelly and Fefferman's work in [8].

If $\|V\|_{L^\infty} \leq K_0$ for some large $K_0 > 1$, Bourgain and Kenig [4] considered the vanishing order for (1.3) with the background from Anderson localization for the Bernoulli model. Bourgain and Kenig established that

$$\|u\|_{L^\infty(\mathbb{B}_r)} \geq c_1 r^{c_2 K_0^{\frac{2}{3}}} \qquad \text{as } r \to 0, \tag{1.4}$$

where c_1, c_2 depend only on n and upper bound of the solution. The estimates (1.4) imply that the upper bound order of vanishing for solutions is less than $CK_0^{\frac{2}{3}}$. Moreover, Kenig in [16] pointed out that the exponent $\frac{2}{3}$ of $K_0^{\frac{2}{3}}$ is optimal for complex-valued potential function $V(x)$ based on Meshkov's example in [23]. Especially, from the example of classic eigenfunctions, Kenig asked if the upper bound of the vanishing order is $CK_0^{\frac{1}{2}}$ for real-valued potential function $V(x)$, which is closely related to Landis' conjecture.