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Rates of convergence to equilibrium for collisionless kinetic equations in slab geometry

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This work deals with free transport equations with partly diffuse stochastic boundary operators in slab geometry. Such equations are governed by stochastic semigroups in L^1 spaces. We prove convergence to equilibrium at the rate $O\left(t^{-\frac{k}{2(k+1)+1}}\right)$ $(t \to +\infty)$ for L^1 initial data g in a suitable subspace of the domain of the generator T where $k \in \mathbb{N}$ depends on the properties of the boundary operators near the tangential velocities to the slab. This result is derived from a quantified version of Ingham's tauberian theorem by showing that $F_g(s) := \lim_{\varepsilon \to 0_+} (is + \varepsilon - T)^{-1} g$ exists as a C^k function on $\mathbb{R} \setminus \{0\}$ such that $\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq \frac{C}{|s|^{2(j+1)}}$ near $s = 0$ and bounded as $|s| \to \infty$ $(0 \leq j \leq k)$. Various preliminary results of independent interest are given and some related open problems are pointed out.

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1. Introduction

This paper is devoted to rates of convergence to equilibrium for one-dimensional free (i.e. collisionless) transport equations with mass-preserving partly diffuse boundary operators. We provide a general L^1 theory relying on a *quantified tauberian theorem* [11]. In linear or non-linear kinetic theory, various non-local (combinations of specular and diffuse) boundary conditions are physically relevant, see e.g. [14][23] and the references therein. Furthermore, general free transport equations with smooth vector fields and positive contractive boundary operators are well posed, see e.g. [4][5]. On the other hand, the existence of an invariant density and the return to this equilibrium state for solutions to free transport equations has not received much attention; see however [1][3][15][27] for the vector field $v.\nabla_x$ with a Maxwell diffuse boundary operator with constant temperature; in this case, the invariant density is given by a maxwellian function. The L^1 convergence to this maxwellian equilibrium goes back to [3] while the analysis of rates of convergence was considered more recently in [1][15] after some numerical investigations in [27]; we will comment below on some results in [1][15]. We note that collisionless transport semigroups present a lack of spectral gap which make them akin to collisional linear kinetic equations with *soft* potentials. More recently, the authors of [20] provided a convergence theory to equilibrium for a general class of monoenergetic free transport equations in slab geometry with azimuthal symmetry and abstract boundary operators. In this abstract model, the existence of invariant density is characterized and shown for a general class of partly diffuse boundary operators. Our aim here is to derive a quantified version (with algebraic rates) of this convergence theory from a quantified version of Ingham's tauberian theorem [11]. We provide a general theory based on some natural structural conditions on the boundary operators in the vicinity of the tangential velocities to the slab. To keep the ideas of this work more transparent, we restrict ourselves to monoenergetic models; (non-monoenergetic free models in slab geometry could be treated similarly, see Remark 33). Besides the main result on the rates of convergence, our construction provides us with various new mathematical results of independent interest. Several open problems are also pointed out.

We note that a special quantified version of Ingham's theorem for "asymptotically analytic" C_0-semigroups (see [11] Corollary 2.12) was already used for the first time in kinetic theory to deal with spatially homogeneous linear Boltzmann equations with soft potentials where the generators are bounded [18]. Finally, we point out that there exists a substantial literature on rates of convergence to equilibrium for collisional (linear or non-linear) kinetic equations relying mostly on *entropy methods*. In particular, collisional kinetic equations with soft potentials exhibit algebraic rates of convergence, see e.g. [6][7][12][18][26] and references therein.

We consider here the monoenergetic free transport equation in slab geometry with azimuthal symmetry

$$\frac{\partial f}{\partial t}(t,x,v)+v\frac{\partial f}{\partial x}(t,x,v)=0, \quad (x,v)\in\Omega \tag{1}$$