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Upper bounds for L^q norms of Dirichlet polynomials with small q

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ABSTRACT

We improve on previous upper bounds for the qth norm of the partial sums of the Riemann zeta function on the half line when $0 < q \leqslant 1$. In particular, we show that the 1-norm is bounded above by $(\log N)^{1/4}(\log\log N)^{1/4}$.

1. Introduction and statement of results

The L^q norm of a Dirichlet polynomial $F(s) = \sum_{n\leqslant N} a(n)n^{-s}$ is defined as

$$\|F\|_q^q = \lim_{T\to\infty} \frac{1}{T}\int_0^T |F(it)|^q dt$$

for $0 < q < \infty$. In this note we are interested in the norms of the Dirichlet polynomial

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$$f(s) = \sum_{n \leqslant N} \frac{1}{n^{1/2+s}}.$$

The norms of this function share similarities with moments of the Riemann zeta function on the half-line, being sometimes referred to as pseudomoments of the Riemann zeta function [1,3]. In the paper [3], Conrey and Gamburd showed that for even $k \in \mathbb{N}$

$$\|f\|_k \sim c_k (\log N)^{k/4} \tag{1}$$

as $N \to \infty$ for a specific constant c_k. This of course bears a strong resemblance to the Keating–Snaith conjecture for the Riemann zeta function [11] which states that

$$\left(\frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2} + it)|^k dt \right)^{1/k} \sim C_k (\log T)^{k/4}, \qquad k > -1$$

as $T \to \infty$ where C_k is some specific constant (slightly different from c_k). As is well known, Keating and Snaith based their conjecture on a formula for the average of the characteristic polynomial over the unitary group. Interestingly, Conrey and Gamburd's proof of (1) exhibits a rigorous connection with random matrix theory.

For the continuous case $0 < q < \infty$, the norms of f were investigated by Bondarenko, the author, and Seip in [2]. It was shown that when $q > 1$, we have the same order of magnitude as Conrey and Gamburd's result i.e.

$$\|f\|_q \asymp (\log N)^{q/4}, \qquad q > 1 \tag{2}$$

and that the lower bound here holds for all $q > 0$, that is

$$\|f\|_q \gg (\log N)^{q/4}, \qquad q > 0.$$

The situation regarding upper bounds for $q \leqslant 1$ is less satisfactory and somewhat more interesting. Here, the results of [2] gave

$$\|f\|_q \ll \begin{cases} (\log N)^{1/4} \log\log N & q = 1 \\ (\log N)^{1/4} & q < 1. \end{cases} \tag{3}$$

The proof of these upper bounds relied on estimates for the norm of the partial sum operator S_N which is defined by

$$S_N\left(\sum_{n=1}^{\infty} a(n) n^{-s} \right) = \sum_{n \leqslant N} a(n) n^{-s}.$$

On applying a generalisation of M. Riesz' Theorem due to Helson [8], it can be shown that for Dirichlet series $F(s)$ in $\mathscr{H}^q$ we have