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Convexity and star-shapedness of matricial range

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ABSTRACT

Let $\mathbf{A} = (A_1, \ldots, A_m)$ be an m-tuple of bounded linear operators acting on a Hilbert space $\mathcal{H}$. Their joint (p, q)-matricial range $\Lambda_{p,q}(\mathbf{A})$ is the collection of $(B_1, \ldots, B_m) \in \mathbf{M}_q^m$, where $I_p \otimes B_j$ is a compression of A_j on a pq-dimensional subspace. This definition covers various kinds of generalized numerical ranges for different values of p, q, m. In this paper, it is shown that $\Lambda_{p,q}(\mathbf{A})$ is star-shaped if the dimension of $\mathcal{H}$ is sufficiently large. If $\dim \mathcal{H}$ is infinite, we extend the definition of $\Lambda_{p,q}(\mathbf{A})$ to $\Lambda_{\infty,q}(\mathbf{A})$ consisting of $(B_1, \ldots, B_m) \in \mathbf{M}_q^m$ such that $I_\infty \otimes B_j$ is a compression of A_j on a closed subspace of $\mathcal{H}$, and consider the joint essential (p, q)-matricial range

$$\Lambda_{p,q}^{ess}(\mathbf{A}) = \bigcap \{\mathbf{cl}(\Lambda_{p,q}(A_1 + F_1, \ldots, A_m + F_m)) : F_1, \ldots, F_m \text{ are compact operators}\}.$$

Both sets are shown to be convex, and the latter one is always non-empty and compact.

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1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. If $\mathcal{H}$ has dimension $n < \infty$, we identify $\mathcal{B}(\mathcal{H})$ with $\mathbf{M}_n$, the space of $n \times n$ complex matrices. The numerical range of $A \in \mathcal{B}(\mathcal{H})$ is defined and denoted by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H},\ \|x\| = 1\}.$$

It is a useful concept for studying matrices and operators; see [10,13]. The Toeplitz–Hausdorff Theorem asserts that this set is always convex [12,23], i.e. $tw_1 + (1-t)w_2 \in W(A)$ for all $w_1,\ w_2 \in W(A)$ and $0 \le t \le 1$. As shown by many researchers, there are interesting interplay between the geometrical properties of the numerical ranges and the algebraic and analytic properties of the operators; for example; see [1,10,11,13]. Motivated by problems from theoretical and applied areas, researchers have considered different generalizations of the numerical range, and extended the results on the classical numerical range to the generalized numerical ranges. We mention a few of them related to our study in the following.

Let $\mathcal{V}_q$ denote the set of operators $X : \mathcal{K} \to \mathcal{H}$ for some q-dimensional subspace $\mathcal{K}$ of $\mathcal{H}$ such that $X^*X = I_{\mathcal{K}}$. To study the compressions of $A \in \mathcal{B}(\mathcal{H})$ on a subspace of $\mathcal{H}$, researchers consider the q-matricial range defined by

$$W(q : A) = \{X^*AX : X \in \mathcal{V}_q\} \subseteq \mathbf{M}_q.$$

One may see the basic references [19,22,24] and the excellent survey [9] on the topic. We remark that $W(q : A)$ is called spatial matricial range in [9].

In the study of joint behavior of several operators in $\mathcal{B}(\mathcal{H})$, researchers consider the joint numerical range of an m-tuple $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$,

$$W(\mathbf{A}) = \{(\langle A_1x, x\rangle, \ldots, \langle A_mx, x\rangle) : x \in \mathcal{H},\ \|x\| = 1\}.$$

In the study of control theory, this is known as the m-multiform numerical range, and the convexity of the sets is useful; see [3,8,14] and their references.

In connection to the study of quantum error correction, researchers study the (p, q)-matricial range $\Lambda_{p,q}(A)$ of $A \in \mathcal{B}(\mathcal{H})$ defined as follows. Let p, q be positive integers with $pq \le \dim \mathcal{H}$. Then

$$\Lambda_{p,q}(A) = \{B \in \mathbf{M}_q : X^*AX = I_p \otimes B \text{ for some } X \in \mathcal{V}_{pq}\}.$$

When $q = 1$, the definition reduces to the rank p-numerical range of A defined by

$$\Lambda_p(A) = \{b : X^*AX = bI_p \text{ for some } X \in \mathcal{V}_p\}.$$