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Grünbaum's inequality for sections

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We show

$$\frac{\int_{E\cap\theta^+} f(x)dx}{\int_E f(x)dx} \geq \left(\frac{k\gamma+1}{(n+1)\gamma+1}\right)^{\frac{k\gamma+1}{\gamma}}$$

for all k-dimensional subspaces $E \subset \mathbb{R}^n$, $\theta \in E \cap S^{n-1}$, and all γ-concave functions $f : \mathbb{R}^n \to [0,\infty)$ with $\gamma > 0$, $0 < \int_{\mathbb{R}^n} f(x)\,dx < \infty$, and $\int_{\mathbb{R}^n} xf(x)\,dx$ at the origin $o \in \mathbb{R}^n$. Here, $\theta^+ := \{x : \langle x, \theta\rangle \geq 0\}$. As a consequence of this result, we get the following generalization of Grünbaum's inequality:

$$\frac{\mathrm{vol}_k(K \cap E \cap \theta^+)}{\mathrm{vol}_k(K \cap E)} \geq \left(\frac{k}{n+1}\right)^k$$

for all convex bodies $K \subset \mathbb{R}^n$ with centroid at the origin, k-dimensional subspaces $E \subset \mathbb{R}^n$, and $\theta \in E \cap S^{n-1}$. The lower bounds in both of our inequalities are the best possible, and we discuss the equality conditions.

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1. Introduction

An elegant inequality of Grünbaum [5] gives a lower bound for the volume of that portion of a convex body lying in a halfspace which slices the convex body through its centroid. Let K be a *convex body* in $\mathbb{R}^n$; that is, a convex and compact set with non-empty interior. Assume that the *centroid* of K,

$$g(K) := \frac{1}{\mathrm{vol}_n(K)} \int_K x\, dx \in \mathrm{int}(K),$$

is at the origin. Given a unit vector $\theta \in S^{n-1}$, we define $\theta^+ := \{x : \langle x, \theta \rangle \geq 0\}$. Specifically, Grünbaum's inequality states that

$$\frac{\mathrm{vol}_n(K \cap \theta^+)}{\mathrm{vol}_n(K)} \geq \left(\frac{n}{n+1}\right)^n. \tag{1}$$

There is equality when, for example, K is the cone

$$\mathrm{conv}\left(\frac{-1}{n+1}\theta + B_2^{n-1}, \frac{n}{n+1}\theta\right)$$

and B_2^{n-1} is the unit ball in $\theta^\perp$. This volume inequality was independently proven in [7].

Compare Grünbaum's inequality with the following long known lower bound for the distance between the centroid $g(K) = o$ and a supporting hyperplane of K. The *support function* of K is defined by $h_K(x) = \max_{y \in K} \langle x, y \rangle$ for $x \in \mathbb{R}^n$. Evaluated at the unit vector θ, $h_K(\theta)$ gives the distance from the origin to the supporting hyperplane of K in the direction θ. Now, the aforementioned inequality is

$$\frac{h_K(\theta)}{h_K(-\theta) + h_K(\theta)} \geq \frac{1}{n+1}. \tag{2}$$

There is equality when, for example, K is the cone

$$\mathrm{conv}\left(\frac{-n}{n+1}\theta + B_2^{n-1}, \frac{1}{n+1}\theta\right).$$

Refer to pages 57–58 of [1] for a discussion of (2).

A generalization of Grünbaum's inequality was recently established in [8] for projections of a convex body. Let E be a k-dimensional subspace of $\mathbb{R}^n$, $1 \leq k \leq n$, and let $K|E$ denote the orthogonal projection of K onto E. "Grünbaum's inequality for projections" states

$$\frac{\mathrm{vol}_k\Big((K|E) \cap \theta^+\Big)}{\mathrm{vol}_k(K|E)} \geq \left(\frac{k}{n+1}\right)^k. \tag{3}$$