The butterfly sequence: the second difference sequence of the numbers of integer partitions with distinct parts

Cristiano Husu

PII: S0022-314X(18)30148-3
DOI: https://doi.org/10.1016/j.jnt.2018.05.005
Reference: YJNTH 6048

To appear in: *Journal of Number Theory*

Received date: 5 May 2017
Revised date: 30 April 2018
Accepted date: 3 May 2018

YJNTH 6048 JNT-D-17-00341R6

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Cristiano Husu, University of Connecticut

To B. and W.

Abstract

We interpret the second difference of the sequence of the number of strict partitions, for $n \geq 5$, as the sequence of the strict partitions of n with *at least three parts*, *the three largest parts consecutive*, and *the smallest part at least two*. The name *butterfly* describes both a sequence's interpretation and a bijection between subsets of strict partitions. Using cyclotomic polynomials, we compute generating function identities of the butterfly sequence and related sequences both as infinite products and as series filtered by the number of parts. We offer a merging and splitting construction of the butterfly sequence as a sequence of partitions with odd parts larger or equal to 3, and we interpret the butterfly sequence as a sequence of *generalized pentagonal*, *pentagonal with domino*, and *non-pentagonal butterfly partitions*. Euler's Pentagonal Number Theorem and a similar specialization of the Jacobi Triple Product lead to recursive algorithms to compute the butterfly sequence and related sequences.

INTRODUCTION

The present paper is based on previous computations of the Jacobi identity for vertex operator algebras and generating function identities for affine Lie algebras in [H2]. The main object, the *butterfly sequence,* defined as the second difference of the sequence of the number of integer partitions with distinct parts, came to the attention of the author while comparing two very specific combinatorial structures: the structure of level 2 standard representations of the affine Lie algebra $A_1^{(1)}$ (see the last two sections of [LW1]), directly related to the butterfly sequence as the Weyl character formula, for one of these representations, gives Euler's identities for integer partitions with distinct *even* parts, and the structure of the Jacobi identity for relative twisted vertex operators multiplied and divided by cyclotomic polynomials. These structures are, in turn, based on more general work [DL1], [DL2], [H1] and [H2], and on the Z-operator construction, in [LW1] and [LW2], of the standard modules of affine Lie algebras. Similar computations occur in the above structures. For the vertex operator algebras and modules, the Jacobi identity is manipulated via the *main algebraic property of the Dirac delta function* $f(x)\delta(x) = f(1)\delta(x)$, for $f(x)$ in $\mathbb{C}\left[[x]\right]$, and $\delta(x) = \sum_{m=-\infty}^{\infty} x^m = \frac{1}{1-x} + \frac{\frac{1}{x}}{1-\frac{1}{x}}$. For affine Lie algebras and modules, *commutators and anticommutators of generating functions* are computed from the Jacobi identity and by multiplication (and division) by the cyclotomic polynomials $\Phi_1(x) = 1 - x$, $\Phi_2(x) = 1 + x + x^2$, $\Phi_3(x) = 1 + x = \Phi_1(-x)$, and $\Phi_6(x) = 1 + x + x^2 = \Phi_2(-x)$ and extractions of residues. For combinatorial applications, *Euler's distinct even parts and distinct odd parts identities,* the *Rogers-Ramanujan identities* (including their *domino* version), and the *Capparelli identities* (see [C1], [C2] and [TX]) are computed as the principal character of level 2 and 3