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Euler characteristic and Akashi series for Selmer groups over global function fields

Andrea Bandini, Maria Valentino [*,1]

Università degli Studi di Parma, Dipartimento di Matematica, Parco Area delle Scienze, 53/A, 43124 Parma, Italy

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ABSTRACT

Let A be an abelian variety defined over a global function field F of positive characteristic p and let K/F be a p-adic Lie extension with Galois group G. We provide a formula for the (truncated) Euler characteristic $\chi(G, Sel_A(K)_p)$ of the p-part of the Selmer group of A over K. In the special case $G \simeq \mathbb{Z}_p^d$ and A a constant ordinary variety, using Akashi series, we show how the Euler characteristic of the dual of $Sel_A(K)_p$ is related to special values of a p-adic $\mathcal{L}$-function.

1. Introduction

Let $p \in \mathbb{Z}$ be a prime and let G be a profinite p-adic Lie group of finite dimension $d \geqslant 1$ and without elements of order p. Let M be an abelian group and a left G-module and consider the following properties

* Corresponding author.
E-mail addresses: andrea.bandini@unipr.it (A. Bandini), maria.valentino@unipr.it (M. Valentino).
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1. $H^i(G, M)$ is finite for any $i \geqslant 0$;
2. $H^i(G, M) = 0$ for all but finitely many i

where the $H^i(\cdot, \cdot)$ are continuous cohomology groups.

Definition 1.1. If a G-module M verifies **1** and **2**, the *Euler characteristic* of M is defined as

$$\chi(G, M) := \prod_{i \geqslant 0} |H^i(G, M)|^{(-1)^i} .$$

We will use the following notation for the *n-th Euler characteristic*

$$\chi^{(n)}(G, M) := \prod_{i \geqslant n} |H^i(G, M)|^{(-1)^i} .$$

Several examples of computations of Euler characteristic are known, here we point out a couple of them which will have a role in what follows.

(1) Let F be a number field and A/F be an abelian variety, denote by $A[p^\infty]$ the Galois module of p-torsion points of A for $p \in \mathbb{Z}$ a prime. Assume that $G = \mathrm{Gal}(F(A[p^\infty])/F)$ has no element of order p, then by [9, Theorem 5]

$$\chi(G, A[p^\infty]) = 1 .$$

(2) Let F be a global function field of characteristic $p > 0$ and E/F an elliptic curve. For $\ell \neq p$ let F_∞ be the extension generated by the ℓ-power torsion points $E[\ell^\infty]$ of E and denote by G the Galois group of the extension F_∞/F. If $\ell \geqslant 5$, then by [19, Theorem III.15]

$$\chi(G, E[\ell^\infty]) = 1$$

(it is a partial function field counterpart of the previous one).
(3) Euler characteristic formulas for Selmer groups of elliptic curves with $G \simeq \mathbb{Z}_p$ are provided, for example, in [10, Chapter 3] and [12, Section 4]. In particular [12, Theorem 4.1] presents a formula involving the constant term of a characteristic element for the Selmer group of the cyclotomic $\mathbb{Z}_p$-extension of a number field, hence (at least conjecturally via the Main Conjecture of Iwasawa theory for that case) a link between the Euler characteristic of that Selmer group and the special value of the L-function of the elliptic curve (as detailed in [12, p. 91]).

Denote by $\Lambda(G) = \mathbb{Z}_p[[G]] := \varprojlim_U \mathbb{Z}_p[G/U]$ the Iwasawa algebra associated to G, where the limit is taken on the open normal subgroups of G. Let $\mathfrak{M}_H(G)$ be the category of