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Scattering for general-type Dirac systems on the semi-axis: reflection coefficients and Weyl functions

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Abstract

We show that for general-type self-adjoint and skew-self-adjoint Dirac systems on the semi-axis Weyl functions are unique analytic extensions of the reflection coefficients. New results on the extension of the Weyl functions to the real axis and on the existence (in the skew-self-adjoint case) of the Weyl functions follow. Important procedures to recover general-type Dirac systems from the Weyl functions are applied to the recovery of Dirac systems from the reflection coefficients. We explicitly recover Dirac systems from the rational reflection coefficients as well.

MSC: 34A55; 34B20; 34L25; 34A05

Keywords: Reflection coefficient; Weyl function; Jost solution; Self-adjoint Dirac system; Skew-self-adjoint Dirac system; Inverse problem

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1. Introduction

Scattering problems on the axis have been actively studied both in this and in the previous centuries and the corresponding results are crucial, in particular, in the inverse scattering transform method (see, e.g., various important references in [1,3,5,6,10,11,13,15,16,18,37,41]). The scattering problems on the semi-axis are of essential interest in theory and applications as well (see, e.g., [2,4,7,25–31,33,35,47,48,50,52]). At the same time, Weyl–Titchmarsh theory on the semi-axis and finite intervals has been actively investigated (see the books [32,34,45,49], recent

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papers [8,12,42,51] and numerous references therein). It is of interest that Weyl functions are successfully used in the research on the initial-boundary value problems for integrable equations whereas inverse scattering transform is a classical tool for solving Cauchy problems for these equations. All the said above indicates that the scattering and Weyl–Titchmarsh theories and their interconnections are quite important.

In particular, scattering and Weyl–Titchmarsh problems for Dirac systems

$$y'(x,z) = \mathrm{i}\big(zj + jV(x)\big)y(x,z), \quad x \geq 0, \tag{1.1}$$

where $y' := \frac{d}{dx}y$,

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ \breve{v} & 0 \end{bmatrix} \quad (m_1 + m_2 =: m), \tag{1.2}$$

and $m_1, m_2 \in \mathbb{N}$, play an important role. Here I_{m_i} is the $m_i \times m_i$ identity matrix, $\mathbb{N}$ stands for the set of natural numbers, $v(x)$ is an $m_1 \times m_2$ matrix function and $\breve{v}(x)$ is an $m_2 \times m_1$ matrix function.

The most interesting cases are the cases of the self-adjoint Dirac systems

$$y'(x,z) = \mathrm{i}\big(zj + jV(x)\big)y(x,z), \quad \breve{v}(x) = v(x)^* \quad (x \geq 0), \tag{1.3}$$

and of the skew-self-adjoint Dirac systems

$$y'(x,z) = \mathrm{i}\big(zj + jV(x)\big)y(x,z), \quad \breve{v}(x) = -v(x)^* \quad (x \geq 0). \tag{1.4}$$

The Weyl–Titchmarsh theory of the self-adjoint Dirac systems is well-studied (see the references above). It is also known that Weyl–Titchmarsh (or simply Weyl) functions of the self-adjoint Dirac systems (1.3) on the semi-axis are closely related to the scattering data. See, for instance, simple formulas connecting *rational* Weyl functions and reflection coefficients of systems (1.3) (where $m_1 = m_2$) in [25], or some special cases of the scalar system (1.3) in [7]. Let us mention also the case of the scalar self-adjoint Schrödinger equation (see [7, (1.3)] and references in [7]).

For the Weyl–Titchmarsh theory of the skew-self-adjoint Dirac systems (1.4) we refer to the works [14,22,24,38] and [45, Ch. 3] (see also references therein). A study of the Weyl–Titchmarsh theory for systems (1.4) started much later than for systems (1.3) and, in particular, less is known about the interconnections with the scattering theory.

Therefore, general results on the interconnections between Weyl functions and reflection coefficients, which we obtain and use in this work, are of interest for both scattering and Weyl–Titchmarsh theories. The next section presents short preliminaries on the scattering theory of Dirac systems on the semi-axis, and the main section of the paper is Section 3, where Subsection 3.1 is dedicated to the self-adjoint systems and Subsection 3.2 is dedicated to the skew-self-adjoint systems.

Under condition that the entries of V are summable (integrable) on $\mathbb{R}_+ = [0, \infty)$, that is, they belong $L^1(\mathbb{R}_+)$ or, in other words,

$$V(x) \in L^1_{m\times m}(\mathbb{R}_+), \tag{1.5}$$

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