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On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and supercritical sources ☆

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Abstract

The aim of this paper is to study the problem

$$\begin{cases} u_{tt} - \Delta u + P(x, u_t) = f(x, u) & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ u_{tt} + \partial_\nu u - \Delta_\Gamma u + Q(x, u_t) = g(x, u) & \text{on } (0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with C^1 boundary ($N \ge 2$), $\Gamma = \partial\Omega$, Γ_1 is relatively open on Γ, Δ_Γ denotes the Laplace–Beltrami operator on Γ, ν is the outward normal to Ω, and the terms P and Q represent nonlinear damping terms, while f and g are nonlinear perturbations.

In the paper we establish local and global existence, uniqueness and Hadamard well-posedness results when source terms can be supercritical or super-supercritical.

MSC: 35L05; 35L20; 35D30; 35D35; 35Q74

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1. Introduction and main results

1.1. Presentation of the problem and literature overview

We deal with the evolution problem consisting of the wave equation posed in a bounded regular open subset of $\mathbb{R}^N$, supplied with a second order dynamical boundary condition of hyperbolic type, in presence of interior and/or boundary damping terms and sources. More precisely we consider the initial –and–boundary value problem

$$\begin{cases} u_{tt} - \Delta u + P(x, u_t) = f(x, u) & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ u_{tt} + \partial_\nu u - \Delta_\Gamma u + Q(x, u_t) = g(x, u) & \text{on } (0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{in } \overline{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with C^1 boundary (see [36]). We denote $\Gamma = \partial\Omega$ and we assume $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_1 being relatively open on Γ (or equivalently $\overline{\Gamma_0} = \Gamma_0$). Moreover, denoting by σ the standard Lebesgue hypersurface measure on Γ, we assume that $\sigma(\overline{\Gamma}_0 \cap \overline{\Gamma}_1) = 0$. These properties of Ω, Γ_0 and Γ_1 will be assumed, without further comments, throughout the paper. Moreover $u = u(t, x)$, $t \geq 0$, $x \in \Omega$, $\Delta = \Delta_x$ denotes the Laplace operator with respect to the space variable, while Δ_Γ denotes the Laplace–Beltrami operator on Γ and ν is the outward normal to Ω.

The terms P and Q represent nonlinear damping terms, i.e. $P(x, v)v \geq 0$, $Q(x, v)v \geq 0$, the cases $P \equiv 0$ and $Q \equiv 0$ being specifically allowed, while f and g represent nonlinear source, or sink, terms. The specific assumptions on them will be introduced later on.

Problems with kinetic boundary conditions, that is boundary conditions involving u_{tt} on Γ, or on a part of it, naturally arise in several physical applications. A one dimensional model was studied by several authors to describe transversal small oscillations of an elastic rod with a tip mass on one endpoint, while the other one is pinched. See [3,22,23,37,48,47,51] and also [50] were a piezoelectric stack actuator is modeled.

A two dimensional model introduced in [33] deals with a vibrating membrane of surface density μ, subject to a tension T, both taken constant and normalized here for simplicity. If $u(t, x)$, $x \in \Omega \subset \mathbb{R}^2$ denotes the vertical displacement from the rest state, then (after a standard linear approximation) u satisfies the wave equation $u_{tt} - \Delta u = 0$, $(t, x) \in \mathbb{R} \times \Omega$. Now suppose that a part Γ_0 of the boundary is pinched, while the other part Γ_1 carries a constant linear mass density $m > 0$ and it is subject to a linear tension τ. A practical example of this situation is given by a drumhead with a hole in the interior having a thick border, as common in bass drums. One linearly approximates the force exerted by the membrane on the boundary with $-\partial_\nu u$. The boundary condition thus reads as $mu_{tt} + \partial_\nu u - \tau\Delta_{\Gamma_1} u = 0$. In the quoted paper the case $\Gamma_0 = \emptyset$ and $\tau = 0$ was studied, while here we consider the more realistic case $\Gamma_0 \neq \emptyset$ and $\tau > 0$, with τ and m normalized for simplicity. We would like to mention that this model belongs to a more general class of models of Lagrangian type involving boundary energies, as introduced for example in [28].

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