ARTICLE IN PRESS YJDEQ:9385

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Excluding blowup at zero points of the potential by means of Liouville-type theorems ☆

Jong-Shenq Guo [a], Philippe Souplet [b,*]

[a] *Department of Mathematics, Tamkang University, 151, Yingzhuan Road, Tamsui, New Taipei City 25137, Taiwan*
[b] *Université Paris 13, Sorbonne Paris Cité, Laboratoire Analyse, Géométrie et Applications, CNRS (UMR 7539), 93430 Villetaneuse, France*

Received 11 May 2016

Abstract

We prove a local version of a (global) result of Merle and Zaag about ODE behavior of solutions near blowup points for subcritical nonlinear heat equations. As an application, for the equation $u_t = \Delta u + V(x)f(u)$, we rule out the possibility of blowup at zero points of the potential V for monotone in time solutions when $f(u) \sim u^p$ for large u, both in the Sobolev subcritical case and in the radial case. This solves a problem left open in previous work on the subject. Suitable Liouville-type theorems play a crucial role in the proofs.

MSC: primary 35K55; secondary 35B44, 35B53

Keywords: Blowup; Potential; Liouville-type theorem

☆ This work was partially supported by the Ministry of Science and Technology of Taiwan (ROC) under the grant 102-2115-M-032-003-MY3. The first author would like to thank the support of LAGA – University of Paris 13 / CNRS, where part of this work was done. The second author is grateful for the support of the National Center for Theoretical Sciences at Taipei for his visit to Tamkang University, where part of this work was done.

* Corresponding author.
E-mail addresses: jsguo@mail.tku.edu.tw (J.-S. Guo), souplet@math.univ-paris13.fr (P. Souplet).

https://doi.org/10.1016/j.jde.2018.06.025
0022-0396/

1. Introduction

In this paper, we consider the following semilinear heat equation with spatially dependent coefficient in the nonlinearity:

$$u_t = \Delta u + V(x)f(u), \quad 0 < t < T, \ x \in \Omega. \tag{1.1}$$

In the case when V is a positive constant and $f(u) \sim u^p$ with $p > 1$, the blowup behavior of solutions has received considerable attention in the past decades and a rich variety of phenomena has been discovered (see, e.g., the monograph [29] and the references therein). In the case when the potential is nonnegative and nonconstant, it is a natural question whether or not blowup can occur at zero points of the potential V. Although the answer would intuitively seem to be negative at first sight, it was surprisingly found in [6,13,15,14] to be positive or negative depending on the situation (see Remark 1.2 for details).

The goal of this paper is twofold:

(i) rule out the possibility of blowup at zero points of the potential V for monotone in time solutions of equation (1.1) when $f(u) \sim u^p$ for large u.
(ii) prove a local version of a (global) result of Merle–Zaag [22] about ODE behavior of solutions near blowup points for subcritical nonlinear heat equations. This result, of independent interest, and which seems to be new even in the case $V \equiv 1$, will be an essential ingredient for (i).

Let us now state our general assumptions:

$$\Omega \text{ is a domain of } \mathbb{R}^n, \ T \in (0, \infty), \tag{1.2}$$

$$f : [0, \infty) \to [0, \infty) \text{ is a function of class } C^1, \tag{1.3}$$

$$\lim_{s\to\infty} s^{-p} f(s) = 1 \text{ for some } p > 1, \tag{1.4}$$

$$|f'(s)| \le C(1 + s^{p-1}), \quad s \ge 0, \tag{1.5}$$

$$V : \overline{\Omega} \to [0, \infty) \text{ is a Hölder continuous function.} \tag{1.6}$$

Throughout this article, we set

$$p_S = (n+2)/(n-2)_+, \qquad \alpha = 1/(p-1), \qquad \kappa = \alpha^\alpha$$

and we denote the zeroset of V by

$$\mathcal{V}_0 = \big\{x \in \overline{\Omega}; \ V(x) = 0\big\}.$$

Our first main result rules out the possibility of blowup at zero points of the potential V for monotone in time solutions of (1.1), under suitable assumptions. In fact, the case of the homogeneous Dirichlet problem associated with equation (1.1) with $f(0) = 0$ was completely solved in [15]. The more delicate case $f(0) > 0$ was left as an open problem. We here essentially solve it for subcritical p, under a mild geometric assumption. Actually, the result here is formulated in

Please cite this article in press as: J.-S. Guo, P. Souplet, Excluding blowup at zero points of the potential by means of Liouville-type theorems, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.06.025