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Well-posedness and analyticity of solutions to a water wave problem with viscosity

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Abstract

The water wave problem models the free-surface evolution of an ideal fluid under the influence of gravity and surface tension. The governing equations are a central model in the study of open ocean wave propagation, but they possess a surprisingly difficult and subtle well-posedness theory. In this paper we establish the existence and uniqueness of spatially periodic solutions to the water wave equations augmented with physically inspired viscosity suggested in the recent work of Dias et al. (2008) [16]. As we show, this viscosity (which can be arbitrarily weak) not only delivers an enormously simplified well-posedness theory for the governing equations, but also justifies a greatly stabilized numerical scheme for use in studying solutions of the water wave problem.

MSC: 35Q35; 35R37; 76D03; 76D09; 76D33; 76D45; 35B20; 35B40

Keywords: Water wave problem; Viscous surface water waves; Transformed field expansions; Existence and uniqueness of solutions; Analytic dependence

1. Introduction

One of the central problems in fluid mechanics is the accurate modeling of the free-surface motion of a large body of water (e.g., a lake or an ocean) [25,37,2]. It is not only a problem of

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classical interest [35,4,36,9,22], but also one of present importance due to its role in a number of applications from the formation and movement of sandbars, to the forces generated by waves on open–ocean oil rigs, to the propagation of tsunamis. The "water wave equations" are the most faithful and successful model for this problem, but they have a surprisingly difficult and subtle well-posedness theory [38,39,6,5,26]. We refer the interested reader to these papers, their extensive bibliographies, and the recent collection [8] for the state of the art in the field (in particular, see the chapters by Ambrose [7] and Wu [40]).

Due to the extremely important role of this model, we were inspired to find a new proof of well-posedness which did not rely on the sophisticated technology required in the papers mentioned above. While this has proven elusive, we now demonstrate that if a physically motivated viscosity is added, then a straightforward existence and uniqueness result can be established. For this we imitate the second author's previous program [1] where such a philosophy was pursued for a weakly nonlinear approximation of the full water wave problem.

For the incorporation of viscosity, as in [1] we follow the lead of Dias, Dyachenko, and Zakharov [16] who advocated for, essentially, (2.1) below. These equations describe potential flow with dissipation featuring a dispersion relation which, in the small viscosity limit, corresponds to that presented in the classic book of Lamb [25]. They also admit the fundamental problem with "viscous potential flow," that the modeling assumptions inherent in potential flow are incompatible with the presence of viscosity. However, the model has proven useful and we believe that this is a natural way to add viscosity to the water wave problem with the goals we have in mind. We also note the work of one of the authors and Kakleas [24] who used this approach to build a stabilized numerical scheme to model surface wave propagation in the weakly nonlinear regime. It is our intent to implement such a scheme for the full water wave equations, but delay our description for a future publication.

Before proceeding, we point out that our method of proof is rather different from the standard techniques, e.g., described in [28,1]. Rather than seeking a fixed point of a contraction mapping, we follow the approach of Friedman and Reitich to free boundary problems, more specifically in the contexts of the classical Stefan problem [19] and the capillary drop problem [20]. Friedman and Reitich's method is perturbative in nature, expanding the solution in a Taylor series in a parameter which characterizes the deformation of the free interface from a simple, separable geometry. Their proof uses, very strongly, the unique solvability of the governing equations on a *fixed*, *trivial* domain (using separation of variables) to show that higher order corrections satisfy appropriate bounds which demonstrate the strong convergence of the Taylor series for the solution. The difficulties here are certain algebra properties of the relevant function spaces and trace lemmas; these are different in the current context, but we show that their demonstrations can be extended.

Additionally, we point out that due to the nature of the function spaces we introduce, the conclusion of our theorem is not only the well-posedness of our model of viscous water waves, but also the very strong stability of our solutions. Our function spaces demand *exponential* decay in time with the rate determined by the value of the viscosity. Thus, not only do unique solutions exist, they persist globally in time and decay exponentially fast to zero. We can state all of this rather informally in the following result.

Theorem 1.1. *There exists a unique solution to the water wave problem with viscosity provided that the initial data resides in a Sobolev space akin to* $L_t^2(e^{2\alpha t})H_x^s H_y^s$, $s \geq 4$. *The solutions are analytic with respect to a parameter which measures the deformation of the fluid interface*