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Minimizers of mass critical Hartree energy functionals in bounded domains

Yujin Guo [a], Yong Luo [b,a], Qi Zhang [b,a]

[a] *Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, PR China*

[b] *University of Chinese Academy of Sciences, Beijing 100049, PR China*

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Abstract

We consider L^2-constraint minimizers of the mass critical Hartree energy functional with a trapping potential $V(x)$ in a bounded domain Ω of $\mathbb{R}^4$. We prove that minimizers exist if and only if the parameter $a > 0$ satisfies $a < a^* = \|Q\|_2^2$, where $Q > 0$ is the unique positive solution of $-\Delta u + u - \big(\int_{\mathbb{R}^4} \frac{u^2(y)}{|x-y|^2} dy\big)u = 0$ in $\mathbb{R}^4$. By investigating new analytic methods, the refined limit behavior of minimizers as $a \nearrow a^*$ is analyzed for both cases where all the mass concentrates either at an inner point x_0 of Ω or near the boundary of Ω, depending on whether $V(x)$ attains its flattest global minimum at an inner point x_0 of Ω or not. As a byproduct, we also establish two Gagliardo–Nirenberg type inequalities which are of independent interest.

MSC: 35J60; 35Q40; 46N50

Keywords: Hartree energy functional; Minimizers; Boundary blow up

1. Introduction

The nonlinear Hartree problems often arise in studying the mean-field limit for large systems of non-relativistic bosonic atoms and molecules, in a regime where the number of bosons is very

E-mail addresses: yjguo@wipm.ac.cn (Y. Guo), luoyong.wipm@outlook.com (Y. Luo), zhangqi1516@163.com (Q. Zhang).

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large and however the interaction between them is weak, see [15,23]. We also refer [6,16] and references therein for the related physical motivations of Hartree problems.

In this paper we study the following mass critical Hartree energy functional in a bounded domain $\Omega \subset \mathbb{R}^4$:

$$e(a) := \inf_{\{u \in H_0^1(\Omega),\ \|u\|_2^2 = 1\}} E_a(u), \tag{1.1}$$

where the energy functional $E_a(u)$ is defined by

$$E_a(u) := \int_\Omega \big(|\nabla u(x)|^2 + V(x)|u(x)|^2\big)dx - \frac{a}{2}\int_\Omega\int_\Omega \frac{|u(x)|^2|u(y)|^2}{|x-y|^2}dxdy, \quad a > 0, \tag{1.2}$$

and the trapping potential $V(x) \geq 0$ satisfies

(V_1). $\min_{x\in\bar\Omega} V(x) = 0$ and $V(x) \in C^\alpha(\bar\Omega)$ for some $0 < \alpha < 1$.

Here we assume that the bounded domain Ω has C^1 boundary and satisfies the interior ball condition in the sense that for all $x_0 \in \partial\Omega$, there exists an open ball $B \subset \Omega$ such that $x_0 \in \partial B \cap \partial\Omega$. The problem $e(a)$ is stimulated mainly by the recent series of papers [5,8–10] and references therein, where the mass critical variational problems are studied in the whole space $\mathbb{R}^N$, instead of Ω. *The main purpose of this paper* is to present the first analysis of mass critical variational problems in bounded domains Ω. We should remark that comparing with the existing papers [5,8–10], we need investigate some new analytic approaches to overcome many challenging difficulties appeared in this paper, including the ones appeared in the situation where the mass of minimizers for $e(a)$ concentrates near the boundary of Ω.

We recall that the problem $e(a)$ defined in the whole space $\mathbb{R}^4$ was recently studied in [5], where the authors made full use of the following nonlinear Hartree equation

$$-\Delta u + u - \Big(\int_{\mathbb{R}^4} \frac{|u(y)|^2}{|x-y|^2}dy\Big)u = 0 \quad \text{in } \mathbb{R}^4,\ \ u \in H^1(\mathbb{R}^4). \tag{1.3}$$

It is well-known from [12,17,24] that, up to translations, (1.3) admits a unique positive solution which is radially symmetric about the origin. Such a positive solution of (1.3) is always denoted by $Q = Q(|x|)$ throughout this paper. The following first result of this paper shows that the problem $e(a)$ defined in Ω is also well connected with the above unique positive solution $Q > 0$.

Theorem 1.1. *Suppose $V(x)$ satisfies (V_1), then we have*

1. *If $0 \leq a < a^* = \|Q\|_2^2$, then there exists at least one minimizer for $e(a)$.*
2. *If $a \geq a^*$, then there is no minimizer for $e(a)$.*

Theorem 1.1 gives a complete classification for the existence and non-existence of minimizers for $e(a)$. It is also interesting to note that the existence of a threshold a^* stated in Theorem 1.1 is independent of the trap $V(x)$ and the domain Ω as well. As shown in Section 2, the proof

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