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A class of planar vector fields with homogeneous singular points: Solvability and boundary value problems

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Abstract

This paper deals with the solvability of planar complex vector fields with homogeneous degeneracies. Hölder continuous solutions are obtained via a Cauchy type integral operator associated to the vector field. An associated boundary value problem of Riemann–Hilbert type is also considered.

MSC: primary 35A01, 35F05; secondary 35C15, 35J70

Keywords: Global solvability; Homogeneous singularity; Vector fields; Boundary value problems

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1. Introduction

This paper deals with the solvability of complex vector fields in two variables with homogeneous degeneracies. This work is an extension of [8] and is also motivated by the recent results contained in [3], [4], and [5].

Let $L = A\partial_x + B\partial_y$ be a vector field in $\mathbb{R}^2$, where A and B are $\mathbb{C}$-valued smooth functions in $\mathbb{R}^2\backslash\{0\}$ and homogeneous of degree $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$. Various aspects of the solvability of the equation $Lu = f$ and associated boundary value problems are considered in [8], when f is smooth or real analytic. The focus here is to consider more general function spaces for the function f. These spaces depend on the homogeneity degree of L and on the degrees of degeneracies of L along the characteristic lines. The main ingredient used is in understanding the properties of an associated integral operator

$$T_Z f(x, y) = \frac{1}{2\pi i} \int_\Omega \frac{f(\xi, \eta)}{Z(\xi, \eta) - Z(x, y)}\, d\xi\, d\eta\,,$$

where Z is a first integral of L.

The organization of this paper is as follows. In section 2, we introduce the class of vector fields and construct a first integral in polar coordinates for an associated vector field L_0. Technical lemmas, crucial for the study of $T_Z f$ are proved in section 3. In section 4, we define function spaces $\mathcal{F}(\Omega)$ in a domain $\Omega \subset \mathbb{R} \times \mathbb{S}^1$ that depend on the homogeneity degree and degeneracy degrees of L_0. Then using the Lemmas of section 3, the main properties of T_Z are proved in Theorems 4.1, 4.2, and 4.3. The Riemann–Hilbert boundary value problem $L_0 u = f$ in Ω and $\Re(\overline{\Lambda}\, u) = \varphi$ on $\partial\Omega$ is considered in section 5. We prove in Theorem 5.3 that if $f \in \mathcal{F}(\Omega)$, Λ and φ are Hölder continuous on $\partial\Omega$ with $|\Lambda| = 1$ and φ an $\mathbb{R}$-valued function, then the problem has a Hölder continuous solution, provided that the index of Λ is nonnegative. In the last section, we interpret the results of sections 4 and 5 for the initial vector field L with a singular point at $0 \in \mathbb{R}^2$.

2. A class of vector fields

Let

$$L = A(x, y)\partial_x + B(x, y)\partial_y \tag{2.1}$$

be a complex vector field in $\mathbb{R}^2$, where $A, B \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ are homogeneous functions of degree $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Hence, for every $(x, y) \in \mathbb{R}^2$ and for every $t \in \mathbb{R}^+$,

$$A(tx, ty) = t^\lambda A(x, y) \quad \text{and} \quad B(tx, ty) = t^\lambda B(x, y).$$

The conjugate of L is the vector field

$$\overline{L} = \overline{A(x, y)}\partial_x + \overline{B(x, y)}\partial_y,$$

where $\overline{A}$ and $\overline{B}$ are the complex conjugates of A and B.

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