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J. Differential Equations 265 (2018) 5315–5351

Journal of Differential Equations

www.elsevier.com/locate/jde

Diffusive stability against nonlocalized perturbations of planar wave trains in reaction–diffusion systems

Björn de Rijk [a,*], Björn Sandstede [b,1]

[a] *Institut für Analysis, Dynamik und Modellierung, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany*

[b] *Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA*

Received 26 March 2018; revised 27 June 2018

Available online 16 August 2018

Abstract

Planar wave trains are traveling wave solutions whose wave profiles are periodic in one spatial direction and constant in the transverse direction. In this paper, we investigate the stability of planar wave trains in reaction–diffusion systems. We establish nonlinear diffusive stability against perturbations that are bounded along a line in $\mathbb{R}^2$ and decay exponentially in the distance from this line. Our analysis is the first to treat spatially nonlocalized perturbations that do not originate from a phase modulation. We also consider perturbations that are fully localized and establish nonlinear stability with better decay rates, suggesting a trade-off between spatial localization of perturbations and temporal decay rate. Our stability analysis utilizes pointwise estimates to exploit the spatial structure of the perturbations. The nonlocalization of perturbations prevents the use of damping estimates in the nonlinear iteration scheme; instead, we track the perturbed solution in two different coordinate systems.

Keywords: Nonlinear stability; Pointwise estimates; Nonlocalized perturbations; Planar reaction–diffusion systems; Periodic traveling waves

* Corresponding author.

E-mail addresses: bjoern.derijk@mathematik.uni-stuttgart.de (B. de Rijk), bjorn_sandstede@brown.edu (B. Sandstede).

[1] Sandstede was partially supported by the NSF through grant DMS-1714429.

https://doi.org/10.1016/j.jde.2018.07.011

0022-0396/

1. Introduction

In this paper, we investigate the stability of spatially periodic planar traveling waves. Consider a planar reaction–diffusion system of the form

$$u_t = D(u_{xx} + u_{yy}) + f(u), \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0, \quad u \in \mathbb{R}^n, \tag{1.1}$$

where $n \in \mathbb{N}$, $D \in \mathbb{R}^{n\times n}$ is a symmetric, positive-definite matrix, and $f: \mathbb{R}^n \to \mathbb{R}^n$ is a C^2-smooth nonlinearity. We are interested in planar traveling-wave solutions to (1.1) of the form $u(x, y, t) = u_\infty(kx - \omega t)$, where the profile $u_\infty(\zeta)$ is periodic in ζ with period 1, $k \in \mathbb{R}$ denotes the spatial wave number, and $\omega \in \mathbb{R}$ is the temporal frequency of the traveling wave. From now on, we use the term *wave train* to refer to spatially-periodic traveling waves. We note that the terms "rolls" and "stripes" are also used in literature to refer to planar wave trains.

Our goal is to determine whether, and in what sense, the planar wave train $u(x, y, t) = u_\infty(kx - \omega t)$ is stable under perturbations of the initial condition $u(x, y, 0) = u_\infty(kx)$. Part of our motivation stems from the case of planar spiral waves that resemble planar wave trains in the far field: understanding the stability of wave-train solutions to (1.1) is a first step towards any nonlinear stability analysis of planar spiral waves.

Before discussing the nonlinear stability of wave trains for the planar system (1.1), we review the relevant results for the spatially one-dimensional case. Note that the function $u(x, t) = u_\infty(kx - \omega t)$ is also a wave-train solution to the one-dimensional version

$$u_t = Du_{xx} + f(u), \quad x \in \mathbb{R}, \quad t \geq 0, \quad u \in \mathbb{R}^n, \tag{1.2}$$

of (1.1). Throughout, we will assume that the wave train is spectrally stable and refer to §2.1 for details on what this assumption entails. We then consider initial conditions of the form

$$\tilde{u}(x, 0) = u_\infty(kx + \varphi_0(x)) + v_0(x), \qquad \varphi_0(x) \to \varphi_\pm \text{ as } x \to \pm\infty, \tag{1.3}$$

where the perturbation v_0 is sufficiently small in an appropriate function space, so that we change the phase, but not the wave number, of the wave train at time $t = 0$. Let $\tilde{u}(x, t)$ denote the associated solution to (1.2): we may then ask whether $\tilde{u}(x, t)$ converges in an appropriate sense to $u_\infty(kx - \omega t)$, or a translate, as time t goes to infinity.

More generally, we can attempt to write the solution in the form

$$\tilde{u}(x, t) = u_\infty(kx + \varphi(x, t) - \omega t) + \text{ terms that decay at least pointwise in time.} \tag{1.4}$$

For the case $|\varphi_+ - \varphi_-| \ll 1$, it was shown in [23] that (1.4) holds for a function $\varphi(x, t)$ that has an asymptotically self-similar profile as $t \to \infty$: indeed, $\varphi(x, t)$ converges to a moving Gaussian if $\varphi_+ = \varphi_-$ and to a moving error function with amplitude $\varphi_+ - \varphi_-$ in the case where $0 < |\varphi_+ - \varphi_-| \ll 1$. Similar results, though without the explicit asymptotics, were also shown in [9, 13–15,24] using different methods – see Remark 1.1 below for more details. The results in [9] were complemented with explicit asymptotics in [10], recovering the results of [23] for the case $0 < |\varphi_+ - \varphi_-| \ll 1$. The restriction that $|\varphi_+ - \varphi_-|$ is small was recently removed in [8]. We emphasize that, although the initial phase off-set φ_0 can be nonlocalized, the perturbation v_0 in (1.3) has to be localized in all the aforementioned papers, that is, we need to assume that $v_0(x) \to 0$ sufficiently rapidly as $x \to \pm\infty$.