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Global existence of solutions to 2-D Navier–Stokes flow with non-decaying initial data in half-plane ☆

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Abstract

We investigate the Navier–Stokes initial boundary value problem in the half-plane $\mathbb{R}^2_+$ with initial data $u_0 \in L^\infty(\mathbb{R}^2_+) \cap J^2_0(\mathbb{R}^2_+)$ or with non decaying initial data $u_0 \in L^\infty(\mathbb{R}^2_+) \cap J^p_0(\mathbb{R}^2_+)$, $p > 2$. We introduce a technique that allows to solve the two-dimensional problem, further, but not least, it can be also employed to obtain weak solutions, as regards the non decaying initial data, to the three-dimensional Navier–Stokes IBVP. This last result is the first of its kind.

MSC: 35Q30; 76D05; 35K51

Keywords: 2-D Navier–Stokes equations; Non-decaying data; Global solution

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1. Introduction

In this paper we consider the following Navier–Stokes initial boundary value problem

$$\begin{aligned} &u_t + u\cdot\nabla u + \nabla\pi_u = \Delta u\,, \quad \nabla\cdot u = 0 \text{ in } (0,T)\times\Omega\,,\\ &u = 0 \text{ on } (0,T)\times\partial\Omega, \quad u = u_0 \text{ on } \{0\}\times\Omega\,, \end{aligned} \tag{1}$$

where the symbol Ω denotes an exterior domain, $\mathbb{R}^n$ and $\mathbb{R}^n_+$, $n \ge 2$, and by $a\cdot\nabla b$ we mean $(a\cdot\nabla)b$. We look for solutions global in the time to problem (1) with non decaying initial data. The problem of the existence of solutions to (1) with non decaying data has been considered by several authors and, we think that the first results, where $n \ge 2$, go back to the papers [12,14,15,22,23]. But the special case of the two-dimensional problem involves a particular interest for the possibility to obtain global existence in the pointwise norm. A natural setting of the problem is the function space $L^\infty((0,T)\times\Omega)$. In this sense a first result is given by Giga, Matsui and Sawada in [15] limited to the Cauchy problem. Subsequently, in [31] Sawada and Taniuchi improve the L^∞-norm of the solutions of [15]. Based on a result by Zelik in [36], a recent contribute given by Gallay in [9] establishes an estimate that up today is the best one:

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \le c\|u_0\|_{L^\infty(\mathbb{R}^2)}(1 + c\|u_0\|_{L^\infty(\mathbb{R}^2)}t), \text{ for all } t > 0.$$

However all these results concern the Cauchy problem associated to the 2D-Navier–Stokes equations with non decaying data. Subsequently, the problem has been considered in exterior domains. Firstly Abe in [1] gives a result of local existence of the mild solution with initial data $u_0 \in L^\infty(\Omega)$ that can be seen as a weak solution to the Navier–Stokes problem. Then, in [26] Maremonti and Shimizu improve the result by Abe giving the existence and uniqueness of solutions to the Navier–Stokes initial boundary value problem in exterior domains which are defined for all $t > 0$. Actually, these authors are able to prove a smooth extension of the solution determined by Abe. The results contained in [26] can be also seen as a "structure theorem" of the weak solution given in [1]. The result by Maremonti and Shimizu is based on the possibility to reduce the problem to an L^2-theory. In the sense that the solution u is seen as the sum of three fields, that is $u = U + W + w$, where U and W are solutions to a linear problem and keep the non decaying character of the initial data, instead w is the solution to a nonlinear perturbed Navier–Stokes with homogeneous initial data and suitable force data with compact support. For the field w is applicable the L^2-theory (see e.g. [18]). However this approach seems to be unable to work in the case of $\partial\Omega$ not bounded.

More recently, in [2], as particular case of the results by Maremonti and Shimizu, Abe proves global existence in exterior domains by means of the special assumption of $u_0 \in L^\infty(\Omega)$ $(\Omega \subseteq \mathbb{R}^2)$ and $\|\nabla u_0\|_2 < \infty$. In the case of the half-plane, he obtains a result under the assumption that the initial data is decaying from the viewpoint of Hardy's inequality.

Although the geometry of the half-plane, and more in general the one of the half-space, concerns a particular case of the mathematical theory, it is very interesting in the applications and recall the attention of several authors [3,6–8,17]. Therefore the aim of the present paper is to prove that the result obtained by Maremonti and Shimizu in [26] also holds in the half-plane. In order to state our chief results we introduce some notations.

By the symbol $\mathscr{C}_0(\Omega)$, we denote the set of all solenoidal vector fields $\varphi \in C_0^\infty(\Omega)$. By the symbol $J^q(\Omega)$, $q \in (1,\infty)$, we indicate the completion of $\mathscr{C}_0(\Omega)$ in Lebesgue space $L^q(\Omega)$.