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Dynamical behaviors of blowup solutions in trapped quantum gases: Concentration phenomenon

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A R T I C L E I N F O

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A B S T R A C T

The current paper contemplates the blowup dynamics in trapped dipolar quantum gases. More precisely, employing the profile decomposition of bounded sequences in $\dot{H}^1 \cap \dot{H}^{\frac{1}{2}}$, we firstly construct related variational problems and derive two refined Gagliardo–Nirenberg inequalities. Secondly, a compactness lemma is utilized to prove that the blowup solutions with bounded $\dot{H}^{\frac{1}{2}}$ norm and bounded L^3 norm absolutely concentrate at least a fixed amount.

1. Introduction

In the present paper, we investigate several concentration characteristics of the blowup solutions for the following Gross–Pitaevskii equation:

$$\begin{cases} i\varphi_t + \dfrac{1}{2}\triangle\varphi + \beta_1|\varphi|^2\varphi + \beta_2(K * |\varphi|^2)\varphi = 0, \\ \varphi(0,x) = \varphi_0, \quad t \in \mathbb{R}^+, x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $\varphi = \varphi(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{C}$. Here i is the imaginary unit, $\beta_1, \beta_2 \in \mathbb{R}$ are real constants and $\triangle$ is the Laplace operator on $\mathbb{R}^3$. Meanwhile, we denote $*$ by the convolution with respect to x between the local density $|\varphi|^2$ and

$$K(x) = \frac{1 - 3\cos^2\theta}{|x|^3},$$

where $\theta = \theta(x)$ is the angle between a fixed dipole axis $n \in \mathbb{R}^3$ and $x \in \mathbb{R}^3$, such that $|n| = 1$, i.e.

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$$\cos\theta = \frac{x \cdot n}{|x|}.$$

The cubic nonlinearity $|\varphi|^2\varphi$ profiles the common collision between particles, short-range or isotropic, whereas the nonlocal potential $K(x) * |\varphi|^2$ depicts the dipolar interactions. Briefly speaking, Eq. (1) models a new sort of quantum gases in which the particles interact are both non-isotropic and long-range, noted as dipolar Bose–Einstein condensation.

There are numerous literatures devoted to the study of the properties of Eq. (1). In order to better profile the dipolar Bose–Einstein condensation, Yi and You [20] introduced Eq. (1) with a pseudo-potential appropriate to depict particles which interact through long-range dipolar forces and short-range repulsive forces. In other words, Eq. (1) can be recognized as a nonlinear Schrödinger equation which is rescaled in dimensionless form by some mathematical simplification. The existence and uniqueness of ground state as well as the blowup theory are accomplished by Carles, Markowich and Sparber in [4] (also see [12,6,3,2]), and they established some critical results to refer to, such as the Fourier transform of $K(x)$ in $\mathbb{R}^3$ under the condition of $n = (0, 0, 1)$. Based on the above results, Antonelli and Sparber [1] have derived some necessary conditions and a variational formulation to constitute the existence of standing waves of Eq. (1). Moreover, by constructing special cross-constrained invariant sets, Ma and Cao [15] have obtained threshold of global existence and finite time blow up of solutions for Eq. (1). These results are also reflected in [14], the researchers discussed the exact value in cross-constrained problem and concerned the instability of standing waves in the unstable regime.

Further research on the dynamical behaviors of blowup solutions for Eq. (1) is essential. Now we review the known results of the classical nonlinear Schrödinger equation

$$\begin{cases} iu_t + \triangle u + |u|^{p-1}u = 0, \\ u(0,x) = u_0, \quad t \in \mathbb{R}^+, x \in \mathbb{R}^N. \end{cases} \tag{2}$$

The local well-posedness of Cauchy problem (2) is established by Ginibre and Velo [9] in $H^1(\mathbb{R}^N)$. The existence, uniqueness, stability and instability of standing wave solutions, as well as the existence of finite time blowup solutions are investigated (see [17,18,10,19,21]). For $p < 1 + \frac{4}{N}$, Eq. (2) is L^2-subcritical and all solutions are global in $H^1(\mathbb{R}^N)$. It is worth noting that blowup may occur if $p = 1 + \frac{4}{N}$, which is considered as L^2-critical case and the corresponding unique radial positive solution R satisfying the elliptic equation $\triangle R - R + |R|^{\frac{4}{N}} R = 0$. Weinstein [19] proved that the sharp sufficient condition for the existence of blowup solutions in Cauchy problem (2) is $\|u_0\|_{L^2} < \|R\|_{L^2}$. Merle and Tsutsumi [16] showed a phenomenon of L^2-concentration at the origin. Moreover, Hmidi and Keraani [13] proposed a modified version of compactness lemma adjusted to the mass concentration. For the L^2-supercritical case $1 + \frac{4}{N} < p < \frac{4}{N-2} + 1$, Zhu [22] and Guo [11] proved that the blowup solutions with bounded $\dot{H}^{s_c}$ norm and L^{p_c} norm absolutely concentrate at least a fixed amount of $\dot{H}^{s_c}$ norm and L^{p_c} norm, respectively.

In the current paper, we investigate the dynamical behaviors of blowup solutions for Cauchy problem (1), which can be recognized as L^2-supercritical and $\dot{H}^1$-subcritical case. More narrowly, employing the profile decomposition of bounded sequences in $\dot{H}^1 \cap \dot{H}^{\frac{1}{2}}$, we firstly construct related variational structures and derive two refined Gagliardo–Nirenberg inequalities (17) and (18). Secondly, a compactness lemma is utilized to prove that the blowup solutions with bounded $\dot{H}^{\frac{1}{2}}$ norm and bounded L^3 norm absolutely concentrate at least a fixed amount.

Here are some notations: $s_c = \frac{N}{2} - \frac{2}{p-1}$, $p_c = \frac{N(p-1)}{2}$. $H^1(\mathbb{R}^3) = W^{1,2}(\mathbb{R}^3)$ denotes the standard Sobolev space. Moreover, we define the pseudo-differential operator $(-\triangle)^s$ as $\widehat{(-\triangle)^s u}(\xi) \equiv |\xi|^{2s}\hat{u}(\xi)$, which directly defines the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3) := \{u \in \mathcal{S}'(\mathbb{R}^3) : \int |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi < +\infty\}$ with its norm $\|u\|_{\dot{H}^s} = \|(-\triangle)^{\frac{s}{2}} u\|_2$, where $\hat{u}$ represents the Fourier transform of u defined in Section 2, and $\mathcal{S}'$ denotes the dual space of Schwartz space $\mathcal{S}$. For convenience, we denote $L^p(\mathbb{R}^3)$, $H^s(\mathbb{R}^3)$, $\dot{H}^s(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} \cdot dx$ as L^p, H^s, $\dot{H}^s$ and $\int \cdot dx$, respectively.