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Note

On the paper "On an identity for the zeros of Bessel functions" by Baricz et al.

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ABSTRACT

In this note we offer some criticism on the paper "On an identity for zeros of Bessel functions" by Baricz et al. [3]. The paper gives identities of type Stieltjes–Calogero for the sums of reciprocals of differences of fourth powers of zeros of Bessel functions. Although interesting in principle, by containing one too many sums of similar complexity the identities fail to convey the true spirit of the work of Stieltjes and Calogero. We rectify this by providing what we think is the correct type of identity for the above-said sums, in the general setup of entire functions of order < 2.

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Let $f(z)$ be an entire function, and denote by $Z(f)$ the multiset of its zeros (zeros are repeated according to multiplicity). Since the pioneering work of Stieltjes and Calogero [11,12,4,5], for fixed $\xi \in Z(f)$ and positive integers p and q, the evaluation of the sums of type

$$\sum_{\substack{\zeta \in Z(f) \\ \zeta \neq \xi}} \frac{1}{(\zeta^p - \xi^p)^q} \tag{1}$$

abound in the literature [2,1,3,7–9]. Typically, p is small ($p = 1, 2$), and the convergence of such sums is guaranteed by working with entire functions $f(z)$ of appropriate finite order of growth at infinity. In [3], $p = 4$ and $q = 1$, and the zeros come from Bessel functions. Logically, a proper evaluation of (1) should not involve other sums of similar complexity, which is not the case for the paper under scrutiny — as an example, take the identity given in Theorem 1, p. 28:

$$\sum_{n \geq 1, n \neq k} \frac{1}{j_{\nu,n}^4 - j_{\nu,k}^4} = -\frac{1}{2j_{\nu,k}^2} \sum_{n \geq 1} \frac{1}{j_{\nu,n}^2 + j_{\nu,k}^2} + \frac{\nu + 2}{4j_{\nu,k}^2}, \tag{2}$$

where $j_{\nu,n}, n \geq 1$ are the positive zeros of the Bessel function of first kind $J_\nu(z)$, $\nu > -1$ real number.

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In this short note we propose a better alternative for identities similar to (2), in the form of the following

Theorem. *Let* $f(z) = \sum_{n=0}^{\infty} c_n z^{2n}$, $c_n \in \mathbf{C}$, $c_0 \neq 0$, *be an even entire function of order* < 2, *with (finitely or infinitely many) simple zeros* $Z(f) = \{\pm a_n | n = 1, 2, \dots\}$, *and such that* $f(ia_n) \neq 0$, $i = \sqrt{-1}$. *Fix a zero* a_k, $k > 0$. *Then*

$$\sum_{n, n \neq k} \frac{1}{a_n^4 - a_k^4} = \frac{3}{8a_k^4} - \frac{1}{8a_k^3} \frac{f''(a_k)}{f'(a_k)} - \frac{i}{4a_k^3} \frac{f'(ia_k)}{f(ia_k)}. \tag{3}$$

If $g(z) = f(\sqrt{z})$, *i.e.* $g(z)$ *is the entire function with power series expansion* $g(z) = \sum_{n=0}^{\infty} c_n z^n$, *then we also have*

$$\sum_{n, n \neq k} \frac{1}{a_n^4 - a_k^4} = \frac{1}{4a_k^4} - \frac{1}{4a_k^2} \frac{g''(a_k^2)}{g'(a_k^2)} + \frac{1}{2a_k^2} \frac{g'(-a_k^2)}{g(-a_k^2)}. \tag{4}$$

Proof. It is easier to prove the identity (4) first, so we will start with it. Since $g(z^2) = f(z)$, and thus $2zg'(z^2) = f'(z)$, by hypothesis the only zeros of $g(z)$ are a_n^2, $n = 1, 2, \dots$, and since $a_n \neq 0$, they are simple zeros. Also, as $f(z)$ has order < 2, $g(z)$ is an entire function of order < 1. It is then well-known [6] that for $k > 0$ fixed, the sums

$$\sum_{n, n \neq k} \frac{1}{a_n^2 - a_k^2} \quad \text{and} \quad \sum_{n} \frac{1}{a_n^2 + a_k^2} \tag{5}$$

are convergent. (In Equation (5) notice that $a_n^2 + a_k^2 \neq 0$, or else $a_n = \pm i a_k$, and so $0 = f(a_n) = f(\pm i a_k) \neq 0$, a contradiction.) For $n \neq k$ we have

$$\frac{1}{a_n^4 - a_k^4} = \frac{1}{2a_k^2} \left(\frac{1}{a_n^2 - a_k^2} \right) - \frac{1}{2a_k^2} \left(\frac{1}{a_n^2 + a_k^2} \right),$$

and consequently

$$\begin{aligned} \sum_{n, n \neq k} \frac{1}{a_n^4 - a_k^4} &= \frac{1}{2a_k^2} \sum_{n, n \neq k} \frac{1}{a_n^2 - a_k^2} - \frac{1}{2a_k^2} \sum_{n, n \neq k} \frac{1}{a_n^2 + a_k^2} \\ &= \frac{1}{4a_k^4} + \frac{1}{2a_k^2} \sum_{n, n \neq k} \frac{1}{a_n^2 - a_k^2} - \frac{1}{2a_k^2} \sum_{n} \frac{1}{a_n^2 + a_k^2}. \end{aligned} \tag{6}$$

Again, being of order < 1 $g(z)$ admits a Hadamard product representation [6]

$$g(z) = c_0 \prod_n \left(1 - \frac{z}{a_n^2} \right), \tag{7}$$

which yields the following logarithmic derivative for $g(z)$:

$$\frac{g'(z)}{g(z)} = -\sum_n \frac{1}{a_n^2 - z} = -\frac{1}{a_k^2 - z} - \sum_{n, n \neq k} \frac{1}{a_n^2 - z}. \tag{8}$$

In Equation (8) the convergence is uniform and absolute on the compact subsets of $\mathbf{C} \setminus Z(g)$.

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