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# A selected pullback attractor

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We introduce a new class of multivalued processes generated by generalized processes and a new concept of pullback attractor, named a selected pullback attractor, and we establish suitable conditions for the existence of such a selected pullback attractor. Finally, we give and application to nonautonomous problems with dynamic boundary conditions.



## 1. Introduction

In [15], the authors considered multivalued semigroups defined by generalized semiflows and introduced $\varphi$-concepts those which are defined without supposing any uniformity on time, i.e., they added the prefix $\varphi$ to the words attraction, dissipativity, boundedness, etc, in order to indicate that was not supposing that the time in such definitions was uniformly chosen in any sense. For example, they said that a set $A$ $\varphi$-attracts some subset $M$ of the phase space $X$ if given $\epsilon > 0$, each solution in the generalized semiflow starting at some point in $M$, eventually goes into the $\epsilon$ neighborhood of $A$, $O_\epsilon(A)$, and remains inside it. They established some abstract results in order to guarantee the existence of a minimal closed global $\varphi$-attractor.

For some problems it is very difficult or even impossible to guarantee the existence of a global attractor which attracts all the solutions of the problem, this is the case, for example of $3D$ Bénard systems, see for example the works [6,7] where the authors proved the existence of the $\varphi$-attractor for such kind of problems.

The concept of pullback attraction has been used a lot in the last years to deal with nonautonomous problems, which attracts the solutions of the problem from $-\infty$, i.e., the initial time goes to $-\infty$ while the final time remains fixed, see for example [10,11]. For the treatment of nonautonomous equations without

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uniqueness of solution, Ball defined in [1] the generalized process. In [14,16] the authors added to that definition some extra conditions such as concatenation in order to obtain an exact generalized process.

For some non-autonomous problems it is also very difficult or even impossible to guarantee the existence of a pullback attractor which pullback attracts all the solutions of the problem, for this reason we will introduce the selected pullback attractor for a generalized process which will be in some sense the analogous to the $\varphi$-attractor for the non-autonomous case.

The concept of selected pullback attractor is very close to the concept of weak pullback attractor for setvalued process developed in [2] and [9] where the main difference is that the weak pullback attractor pullback attracts bounded sets with time being uniform on the bounded set whereas the selected pullback attractor pullback attracts points without supposing any uniformity on time. In the 1960s E. Roxin distinguished between strong and weak properties for the trajectories generated by a non-autonomous control system, such properties may be for all or at least one trajectory, for example the attraction property. Such theory was investigated by Kloeden in the 1970s, see article [8].

We organize the work as follows. Section 2 contains notations, definitions, and some properties on the multivalued process defined by an exact generalized process. In Sect. 3 we introduce the selected pullback attractor and we establish sufficient conditions for the existence of such attractor. Finally, we give an application in Sect. 4 to nonautonomous equations with dynamic boundary conditions.

## 2. B-pullback asymptotically compact property for generalized processes

In this section we present the main facts about a generalized process and an alternative property to ensure the asymptotic compactness. Several results mentioned in this section are demonstrated in [14].

Let $(X, \rho)$ be a complete metric space. For $x \in X$, $A, B \subset X$ and $\epsilon > 0$ we define

$$\rho(x, A) := \inf_{a \in A}\{\rho(x, a)\};$$

$$dist(A, B) := \sup_{a \in A}\inf_{b \in B}\{\rho(a, b)\};$$

$$\mathcal{O}_\epsilon(A) := \{z \in X; \rho(z, A) < \epsilon\}.$$

Denote by $\mathcal{P}(X)$, $\mathcal{B}(X)$ and $\mathcal{K}(X)$ the nonempty, nonempty and bounded and nonempty and compact subsets of $X$, respectively.

**Definition 1.** A **generalized process** $\mathscr{G} = \{\mathscr{G}(\tau)\}_{\tau\in\mathbb{R}}$ in $X$ is a family of sets $\mathscr{G}(\tau)$ consisting of functions $\varphi : [\tau, \infty) \to X$, called solutions, satisfying the following conditions:

(C1) For each $\tau \in \mathbb{R}$ and $z \in X$ there exists at least one $\varphi \in \mathscr{G}(\tau)$ with $\varphi(\tau) = z$;
(C2) If $\varphi \in \mathscr{G}(\tau)$ and $s \geq 0$, then $\varphi^{+s} \in \mathscr{G}(\tau + s)$, where $\varphi^{+s} := \varphi_{|[\tau+s,\infty)}$;
(C3) If $\{\varphi_j\}_{j\in\mathbb{N}} \subset \mathscr{G}(\tau)$ and $\varphi_j(\tau) \to z$, there is a subsequence $\{\varphi_{j_k}\}_{k\in\mathbb{N}}$ of $\{\varphi_j\}_{j\in\mathbb{N}}$ and $\varphi \in \mathscr{G}(\tau)$ with $\varphi(\tau) = z$ and such that $\varphi_{j_k}(t) \to \varphi(t)$ when $k \to \infty$, for each $t \geq \tau$.

**Definition 2** *(Exact generalized process).* A generalized process $\mathscr{G} = \{\mathscr{G}(\tau)\}_{\tau\in\mathbb{R}}$ is called **exact** if it satisfies:

(C4) (Concatenation) Let $\varphi \in \mathscr{G}(\tau)$ and $\psi \in \mathscr{G}(r)$ such that $\varphi(s) = \psi(s)$ for some $s \geq r \geq \tau$. If $\theta$ is defined by

$$\theta(t) := \begin{cases} \varphi(t), \ t \in [\tau, s], \\ \psi(t), \ t \in (s, \infty), \end{cases}$$

then $\theta \in \mathscr{G}(\tau)$.