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Bifurcation solutions of a free boundary problem modeling tumor growth with angiogenesis ☆

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ABSTRACT

In this paper we study bifurcation solutions from the unique radial solution of a free boundary problem modeling stationary state of tumors with angiogenesis. This model comprises two elliptic equations describing the distribution of the nutrient concentration $\sigma = \sigma(x)$ and the inner pressure $p = p(x)$. Unlike similar tumor models that have been intensively studied in the literature where Dirichlet boundary condition for σ is imposed, in this model the boundary condition for σ is a Robin boundary condition. Existence and uniqueness of a radial solution of this model have been successfully proved in a recently published paper [20]. In this paper we study existence of nonradial solutions by using the bifurcation method. Let $\{\gamma_k\}_{k=2}^{\infty}$ be the sequence of eigenvalues of the linearized problem. We prove that there exists a positive integer $k_* \geqslant 2$ such that in the two dimension case for any $k \geqslant k_*$, γ_k is a bifurcation point, and in the three dimension case for any even $k \geqslant k_*$, γ_k is also a bifurcation point.

1. Introduction

In this paper we study the following free-boundary problem modeling the growth of stationary tumors with angiogenesis:

$$\begin{cases} \triangle\sigma = f(\sigma), & \text{in } \Omega, \\ \triangle p = -g(\sigma), & \text{in } \Omega, \\ \dfrac{\partial\sigma}{\partial \boldsymbol{n}} = \beta(\overline{\sigma} - \sigma), & \text{on } \partial\Omega, \\ p = \gamma\kappa, & \text{on } \partial\Omega, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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Here Ω is a bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ where the tumor occupies, with boundary $\partial\Omega$ unknown, $\sigma = \sigma(x)$ and $p = p(x)$ are two unknown functions respectively representing the nutrient concentration and the inner pressure within the tumor region Ω; f and g are given nonlinear real-valued functions defined on $[0, \infty)$, satisfying some reasonable assumptions (see Conditions (A1)–(A4) below); β and $\overline{\sigma}$ are two given positive constants describing the ability of tumor cells absorbing nutrient from the host tumor tissue via its own vasculature and the constant nutrient concentration of the boundary of tumor, respectively; γ is a positive parameter called *surface tension coefficient*, which measures the surface tension of the tumor surface; κ is the mean curvature of the tumor surface whose sign is designated to be positive when $\partial\Omega$ is convex; $\frac{\partial}{\partial \boldsymbol{n}}$ is the derivative in the direction of outward normal $\boldsymbol{n}$ of $\partial\Omega$.

The above problem is the stationary form of a tumor model with angiogenesis that has been recently studied by the authors in the reference [20]. In the case that f and g are linear functions

$$f(\sigma) = \lambda\sigma \quad \text{and} \quad g(\sigma) = \mu(\sigma - \tilde{\sigma}), \tag{1.2}$$

such tumor model was proposed by Friedman and Lam in [13] as an essential modification to the corresponding model of Byrne and Chaplain [2,3]. The modification is made by considering nutrient supply mechanism of the tumor in a different viewpoint from that of Byrne and Chaplain; see [13] and [20] for details of the biological background. We also refer the reader to see [5] for discussion on biological consideration of extending the linear functions (1.2) to general nonlinear functions.

In [20] we proved that under the assumptions (A1)–(A4) given below, the problem (1.1) has a unique radial solution $(\sigma_s(r), p_{s\gamma}(r), \Omega_s)$ for all $\gamma > 0$, where $\Omega_s = \{x \in \mathbb{R}^n : |x| < R_s\}$ and R_s is a positive number. Later on we shall also regard the triple $(\sigma_s(r), p_{s\gamma}(r), R_s)$ as the radial solution of the problem (1.1). Note that $\sigma_s(r)$ and R_s are independent of γ, but $p_{s\gamma}(r)$ depends on γ; see the proof of Theorem 1.1 of [20]. In this paper we consider nonradial solutions of the problem (1.1). We shall use bifurcation theory to treat this problem by regarding γ as a bifurcation parameter. This means that we regard $(\sigma_s(r), p_{s\gamma}(r), \Omega_s)$ as a trivial solution and proceed to study for what values of γ a nontrivial solution $(\sigma(r, \omega),\ p(r, \omega),\ \Omega)$, or in other words, a nonradial solution, will be bifurcated from this trivial solution. Here and hereafter we always denote $r = |x|$ and $\omega = x/r$ (for $x \neq 0$). Note that in the special case that f and g are the linear functions given by (1.2), bifurcations of the problem (1.1) has been analyzed by Huang, Zhang and Hu [15]. We also refer the reader to see [1,6,8,10–12,14,18,19] for bifurcation analysis of the Byrne and Chaplain tumor model and other related tumor models.

Throughout this paper we assume the following group of conditions for f and g are satisfied:

(A1) $f \in C^\infty[0, \infty)$, and $g \in C^\infty[0, \infty)$;

(A2) $f'(\sigma) > 0$ for any $\sigma \in [0, \infty)$, and $f(0) = 0$;

(A3) $g'(\sigma) > 0$ for any $\sigma \in [0, \infty)$, and $g(\widetilde{\sigma}) = 0$ for some $\widetilde{\sigma} > 0$;

(A4) $\widetilde{\sigma} < \overline{\sigma}$.

Recall that the radial solution $(\sigma_s(r), p_{s\gamma}(r), R_s)$ of the problem (1.1) is the unique solution of the following boundary value problem (cf. [20]);

$$\begin{cases} \sigma_s''(r) + \frac{n-1}{r}\sigma_s'(r) = f(\sigma_s(r)), \quad 0 < r < R_s, \\ p_s''(r) + \frac{n-1}{r}p_s'(r) = -g(\sigma_s(r)), \quad 0 < r < R_s, \\ \sigma_s'(0) = 0, \quad \sigma_s'(R_s) = \beta(\bar{\sigma} - \sigma_s(R_s)), \\ p_s'(0) = 0, \quad p_s(R_s) = \gamma/R_s, \\ p_s'(R_s) = 0. \end{cases} \tag{1.3}$$

In Section 2 below we shall prove that the linearized problem of the model (1.1) at the radial solution $(\sigma_s(r), p_{s\gamma}(r), R_s)$ is as follows:

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