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A two-grid method for incompressible miscible displacement problems by mixed finite element and Eulerian–Lagrangian localized adjoint methods ☆

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ABSTRACT

In this paper, we present a scheme for solving two-dimensional miscible displacement problems using Eulerian–Lagrangian localized adjoint methods and mixed finite element methods. Since only the velocity and not the pressure appears explicitly in the concentration equation, an Eulerian–Lagrangian localized adjoint method is used to solve the concentration equation and a mixed finite element method is used for the pressure equation. To linearize and decouple the mixed-method equations, we use a two-grid algorithm based on the Newton iteration method for this fully discrete problems. First, we solve the original nonlinear equations on the coarse grid, then, we solve the linearized problem on the fine grid using Newton iteration once. It is shown that the coarse grid can be much coarser than the fine grid and achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = \mathcal{O}(h^{1/2})$ in this paper.

1. Introduction

Let Ω be a polygonal domain in $\mathcal{R}^2$. In this paper, we consider the following nonlinear coupled system of equations:

$$\phi\frac{\partial c}{\partial t} + \nabla\cdot(\mathbf{u}c - \mathbf{D}\nabla c) = f(c), \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = q, \tag{1.2}$$

$$\mathbf{u} = -a(c)\nabla p, \tag{1.3}$$

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where $x \in \Omega$, $t \in J = [0,T]$, and $a(c) = a(x,c) = \frac{k(x)}{\mu(c)}$. $c(x,t)$ is the concentration of an invading fluid or a concerned solute/solvent, and $p(x,t)$ and $\mathbf{u}(x,t)$ are the pressure and Darcy velocity of the fluid mixture, respectively. $\phi(x)$ and $k(x)$ are the porosity and the permeability tensor of the medium, respectively, and $f(c)$ may be nonlinear function [16]. $\mathbf{D}(\mathbf{u}) = \phi(x)d_m\mathbf{I} + d_t|\mathbf{u}| + (d_l - d_t)(u_i u_j)^2_{i,j=1}/|\mathbf{u}|$ is the diffusion–dispersion tensor, with d_m, d_t, and d_l being the molecular diffusion and the transverse and longitudinal dispersivities, respectively, and $\mathbf{I}$ is the identity tensor. For convenience, we assume that $\mathbf{D} = \phi d_m \mathbf{I}$ implies only the molecular diffusion and not the dispersion in this paper.

The system is subjected to the boundary and initial conditions:

$$\mathbf{u} \cdot \mathbf{n} = (\mathbf{D}\nabla c) \cdot \mathbf{n} = 0, \quad x \in \partial\Omega,\ t \in J, \tag{1.4}$$

$$c(x,0) = c_0(x), \quad x \in \Omega. \tag{1.5}$$

Mathematical models used to describe porous medium flow processes in petroleum reservoir simulation, groundwater contaminant transport, and other applications lead to coupled systems of time-dependent nonlinear partial differential equations [1,7,18]. Due to the nonlinearity and couplings of these equations, the moving steep fronts present in the solution of these partial differential equations, and the enormous size of field-scale application, the numerical treatment of these systems often encounters severe difficulties. Standard finite difference or finite element methods tend to generate numerical solutions with nonphysical oscillations or numerical dispersion along with spurious grid-orientation effect. Therefore, many papers have studied on the development of improved methods related to these models.

In the past forty years, a variety of numerical techniques have been introduced to obtain better approximations for the miscible displacement of one incompressible fluid by another in a reservoir, such as higher-order Godunov method [2], least-squares mixed finite element method [38], the modified method of characteristic finite element method (MMOC-Galerkin) [17,21], and the local discontinuous Galerkin method [27]. Douglas and his coauthors presented and analyzed a finite element method and mixed finite element method (MFEM) time-stepping procedure in [15,16]. Because of the hyperbolic nature of the concentration equation (1.1), characteristic methods have been investigated extensively in numerical simulations of porous medium flows. Subsequently, a MMOC-MFEM time-procedure was proposed and analyzed in [19,20]. The MMOC symmetrizes and stabilizes the concentration equation, greatly reduces temporal errors, and so allows for large time steps in a simulation without loss of accuracy. However, the MMOC fails to conserve mass and has difficulties in handling general boundary conditions.

We apply an Eulerian–Lagrangian localized adjoint method for the concentration $c(x,t)$. The Eulerian–Lagrangian localized adjoint method (ELLAM) framework [6,25] provides a systematic approach to solve transient advection–diffusion PDEs with general boundary conditions in a mass-conservative manner, while maintaining accuracy and efficiency of Eulerian–Lagrangian methods. One of the major advantages of ELLAM is the ability to incorporate boundary conditions into the approximating equations with no change in the overall approach. Boundary terms arise naturally in the ELLAM formulation, and these terms allow specified boundary conditions to be imposed directly. The other advantage of ELLAM is maintaining mass conservation. Thus, the ELLAM framework overcomes the principal shortcomings of the previous characteristic methods while maintaining their numerical advantages. Numerical experiments show that the ELLAM is very competitive in the context of model transport partial differential equations [30,31] and miscible porous medium flow [3,25,33]. Wang et al. [32] have used ELLAM schemes for advection–diffusion partial differential equations, compressible and incompressible fluid flows, which show that ELLAM schemes often outperform many widely used and well-regarded methods. We refer the reader to [30–32] and references therein for more details.

The numerical behavior of (1.1) depends strongly on the accuracy of the approximation of the velocity $\mathbf{u}$, which is because advection and diffusion dispersion in the concentration equation are governed by Darcy velocity. In this paper, a mixed finite element method is used to solve the pressure equation, which has