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Modified shrinking target problem in beta dynamical systems

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A R T I C L E   I N F O

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A B S T R A C T

In the paper, we investigate a modified version of the shrinking target problem in beta dynamical systems which is induced by the β-transformation $T_\beta : x \to \beta x \pmod 1$. The Hausdorff dimension of the lim sup set

$$W(f,\varphi) = \{x \in [0,1) : |T_\beta^n x - f(x)| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

is determined, where $f : [0,1] \to [0,1]$ is a Lipschitz function and φ is a positive function defined on $\mathbb{N}$. The set $W(f,\varphi)$ can be viewed as a dynamical analogue of the inhomogeneous Diophantine approximation in which the inhomogeneous factor is allowed to vary.

1. Introduction

Diophantine properties of the orbits in a dynamical system study the distributions of the orbits, which can be viewed as a quantitative version of the classic Birkhoff's ergodic theorem. Moreover, it has close connection with the classic Diophantine approximation which studies the distributions of rational numbers.

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system with a consistent metric d. If T is ergodic with respect to the measure μ, Birkhoff's ergodic theorem implies that, for any $x_0 \in X$, almost surely

$$\liminf_{n\to\infty} d(T^n x, x_0) = 0.$$

One can then ask how fast the above liminf tends to zero. More precisely, let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a positive function. What are the metric properties of the set

$$W(x_0,\varphi) := \{x \in X : d(T^n x, x_0) < \varphi(n) \ \text{ for infinitely many } n \in \mathbb{N}\}$$

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in the sense of measure and dimension? The points in $W(x_0, \varphi)$ can be thought of as trajectories which hit a shrinking target infinitely often, which is called the shrinking target problem by Hill and Velani [9].

Another kind of shrinking target problem can be formulated as follows which aims at a quantitative study on the Poincaré's recurrence theorem:

$$R(\varphi) := \{x \in X : d(T^n x, x) < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

The set $R(\varphi)$ can also be viewed as the collection of points whose orbits return a shrinking neighborhood of the initial point infinitely often.

The size of the sets $W(x_0, \varphi)$ and $R(\varphi)$ have been well studied in many dynamical systems. For the background and more results, the reader is referred to [3,4,10,13,15,16]. It should be noticed that in many systems, the two sets share the same dimensional formulae. So, one would like to ask whether we can treat the dimensions of $W(x_0, \varphi)$ and $R(\varphi)$ in a unified way? So, this leads to a modified shrinking target problem, i.e. consider the following set

$$W(f, \varphi) = \{x \in [0, 1) : |T^n x - f(x)| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$

where $f : [0, 1] \to [0, 1]$ is a positive function. This setup can also be viewed as a dynamical analogue of the classical inhomogeneous Diophantine approximation in which the inhomogeneous factor is allowed to vary (see [1,2]). Information on the set $W(x_0, \varphi)$ has been plenty of, for examples, there are Birkhoff's ergodic theorem, the dynamical Borel–Cantelli lemma [6] and the results on Hausdorff dimension [9], etc. However, for the set $W(f, \varphi)$, even when φ is a positive constant function, nothing was known.

In this paper, we focus on the size of $W(f, \varphi)$ when T is the β-transformation $T_\beta : x \to \beta x \pmod 1$. In this special system, the Hausdorff dimension, denoted by $\dim_H$, of the set $W(x_0, \varphi)$ is determined by Shen and Wang [14], see also [5], and the dimension of $R(\varphi)$ is presented in [15] by Tan and Wang. We give a complete characterization on the Hausdorff dimension of $W(f, \varphi)$, so includes the above results as special case. Still, the non-Markov property of β constitutes the main difficulty. This is conquered by a detailed usage on the distribution of so-call full cylinders. Let $f : [0, 1] \to [0, 1]$ be a Lipschitz function. Still write

$$W(f, \varphi) = \{x \in [0, 1) : |T_\beta^n x - f(x)| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Theorem 1.1. *Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a positive function and $f : [0, 1] \to [0, 1]$ a Lipschitz function. Then*

$$\dim_H W(f, \varphi) = \frac{1}{1+\alpha}, \text{ where } \alpha = \liminf_{n\to\infty} \frac{-\log_\beta \varphi(n)}{n}.$$

2. Preliminaries

We begin with a brief account on some basic properties of β-expansions and fixing some notation.

The β-expansion of real numbers was first introduced by Rényi [12], which is given by the following algorithm. For any $\beta > 1$, let

$$T_\beta(0) := 0, \quad T_\beta(x) = \beta x - \lfloor \beta x \rfloor, x \in [0, 1), \tag{2.1}$$

where $\lfloor \xi \rfloor$ is the integer part of $\xi \in \mathbb{R}$. By taking

$$\epsilon_n(x, \beta) = \lfloor \beta T_\beta^{n-1} x \rfloor \in \mathbb{N}$$

recursively for each $n \geq 1$, every $x \in [0, 1)$ can be uniquely expanded into a finite or an infinite sequence