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Derivation of vector-valued complex interpolation scales [☆,☆☆]

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ABSTRACT

We study complex interpolation scales obtained by vector valued amalgamation and the derivations they generate. We study their trivial and singular character and obtain examples showing that the hypotheses in the main theorems of Castillo et al. (2017) [9] are not necessary.

1. Introduction

In this paper we study interpolation scales of vector valued functions, the derivations they induce and some of their basic properties, mainly nontriviality and singularity. Our examples show, in particular, that the hypothesis of the main results in [J.M.F. Castillo, V. Ferenczi and M. Gonzalez, *Singular exact sequences generated by complex interpolation*, Trans. Amer. Math. Soc. 369 (2017) 4671–4708] are not necessary. A special attention is payed to the derivations obtained by amalgamation or fragmentation, in the spirit of the Enflo, Lindenstrauss and Pisier construction. Indeed, the first twisted Hilbert space was obtained by Enflo, Lindenstrauss and Pisier [14]. It has the form $\ell_2(F_n)$ for a specific sequence F_n of finite-dimensional Banach spaces. Even if it is not known whether the Enflo–Lindenstrauss–Pisier space can be obtained by derivation, we will obtain and study fragmented scales whose derived spaces have a similar form. To give just one example, fragmentation of the scale (ℓ_1, ℓ_∞) yields the scale $(\ell_2(\ell_1^n), \ell_2(\ell_\infty^n))$, which generates the derived space $\ell_2(Z_2(n))$, where $Z_2(n)$ is the fragmentation of the Kalton–Peck space. And while the

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Kalton–Peck sequence is strictly singular, the fragmented Kalton–Peck sequence is "strictly non-singular" (we thank F. Cabello for this name).

The general theory yields that an admissible couple (X_0, X_1) of Banach spaces for which complex interpolation at θ yields the space X_θ generates an exact sequence

$$0 \longrightarrow X_\theta \xrightarrow{\ j\ } dX_\theta \xrightarrow{\ q\ } X_\theta \longrightarrow 0 \tag{1.1}$$

The middle space dX_θ in (1.1) is called the derived space of the scale (X_0, X_1) at θ. It is especially interesting when $X_\theta = \ell_2$, in which case the space dX_θ is called a twisted Hilbert space (see below). The exact sequence (1.1) is said to be *trivial* when $j(X_\theta)$ is complemented in dX_θ. The exact sequence is called *singular* when the operator q is strictly singular, which means that its restrictions to infinite dimensional closed subspaces are never an isomorphism.

A drawback in the theory is the scarcity of examples. While it is relatively easy to get $(X, X^*)_{1/2} = \ell_2$, it is rather difficult to calculate the associated derivation and study its properties. The paper [2] presents a complete description of the derivations that appear when considering scales of Lorentz spaces, while the paper [9] performs a thorough study of singular derivations. In this paper we continue the previous work by obtaining new examples of derivations, study their properties and show that the hypotheses in the main theorems of [9] are actually not necessary.

2. Exact sequences, twisted sums and centralizers

A *twisted sum* of two Banach spaces Y and Z is a Banach space X which has a subspace isomorphic to Y with the quotient X/Y isomorphic to Z. An exact sequence $0 \to Y \to X \to Z \to 0$ of Banach spaces and linear continuous operators is a diagram in which the kernel of each arrow coincides with the image of the preceding one. By the open mapping theorem this means that the middle space X is a twisted sum of Y and Z.

A special type of exact sequences appear generated by the complex interpolation method when applied to a pair of spaces as we describe now. A sound background on complex interpolation can be found in [1,17]. Let $\mathbb{S}$ denote the open strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ in the complex plane, and let $\overline{\mathbb{S}}$ be its closure. Given an admissible (i.e., a pair that we assume linear and continuously embedded into a Hausdorff topological vector space W) pair (X_0, X_1) of complex Banach spaces, let $\Sigma = X_0 + X_1$ endowed with the norm $\|x\| = \inf\{\|x_0\|_0 + \|x_1\|_1 : x = x_0 + x_1\}$. We denote by $\mathcal{F} = \mathcal{F}(X_0, X_1)$ the space of functions $g : \overline{\mathbb{S}} \to \Sigma$ satisfying the following conditions:

(1) g is $\|\cdot\|_\Sigma$-bounded and $\|\cdot\|_\Sigma$-continuous on $\overline{\mathbb{S}}$, and $\|\cdot\|_\Sigma$-analytic on $\mathbb{S}$;
(2) $g(it) \in X_0$ for each $t \in \mathbb{R}$, and the map $t \in \mathbb{R} \mapsto g(it) \in X_0$ is bounded and continuous;
(3) $g(it + 1) \in X_1$ for each $t \in \mathbb{R}$, and the map $t \in \mathbb{R} \mapsto g(it + 1) \in X_1$ is bounded and continuous.

The space $\mathcal{F}$ is a Banach space under the norm $\|g\|_\mathcal{F} = \sup\{\|g(j + it)\|_j : j = 0, 1; t \in \mathbb{R}\}$. For $\theta \in [0, 1]$, define the interpolation space

$$X_\theta = (X_0, X_1)_\theta = \{x \in \Sigma : x = g(\theta) \text{ for some } g \in \mathcal{F}\}$$

with the norm $\|x\|_\theta = \inf\{\|g\|_\mathcal{F} : x = g(\theta)\}$. So, if $\delta_\theta : \mathcal{F} \to \Sigma$ denotes the obvious evaluation map $\delta_\theta(f) = f(\theta)$ then $(X_0, X_1)_\theta$ is the quotient of $\mathcal{F}$ by $\ker \delta_\theta$, and thus it is a Banach space. For $0 < \theta < 1$, we will consider the maps $\delta_\theta : \mathcal{F} \to \Sigma$ (evaluation of the function at θ) and $\delta'_\theta : \mathcal{F} \to \Sigma$ (evaluation of the derivative at θ). Let $B : X_\theta \to \mathcal{F}$ be a bounded homogeneous selection for δ_θ and set

$$d_{\delta'_\theta B} X_\theta = \{(y, z) \in \Sigma \times X_\theta \,:\, y - \delta'_\theta B z \in X_\theta\}$$