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Note

# Green's formula for integro-differential operators ☆

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ABSTRACT

In this report we establish Green's formula for an integro-differential operator, and apply it to describe a class of self-adjoint fractional order differential operators. A found symmetric fractional order Caputo–Riemann–Liouville type operator can be considered as a fractional analogue of the classical Sturm–Liouville operator in some sense.



## 1. Introduction

In this note we briefly describe a way to introduce symmetric integro-differential operators. Since this kind of operators, in general, cannot be analysed by using the methods of the classical theory of pseudo-differential operators, we here offer, in some sense, a new approach to investigate spectral properties of integro-differential operators.

In this paper we are interested in studying symmetric fractional order differential operators. In general, fractional operators are not symmetric, and in all researches related to spectral properties only non self-adjoint problems are considered. In the recent manuscript [7] one symmetric fractional order differential operator is described by Klimek and Agrawal in the weighted class of continuous functions. Here, we show self-adjointness of one fractional order differential operator of Caputo–Riemann–Liouville type. Moreover, continue research that is started in [13], where the authors began to describe self-adjoint fractional order differential operators.

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Our main goal is to establish an analogue of the Green's formula for fractional order differential equations, by using techniques offered in the recent papers of Ruzhansky and his co-authors [4,10,11]. As application, we describe a class of self-adjoint operators, and by the duality, we define fractional order differential operators on the space of distributions.

Here, fractional differential operators of the Caputo and Riemann–Liouville type are objects to study. In the end, we describe a class of self-adjoint problems related to this fractional order differential equation in $L^2(0,1)$, and formulate several statements on the spectral properties of the problems. Indeed, it is found a symmetric Caputo–Riemann–Liouville operator of the order $2\alpha$ (with $\frac{1}{2} < \alpha < 1$). In appreciate sense, it can be interpreted as an analogue of the classical Sturm–Liouville operator.

## 2. Fractional differentiation and its properties

Here, we recall definitions and properties of fractional integration and differentiation operators [12,9,6].

**Definition 2.1.** Let $f$ be a function defined on the interval $[0,1]$. Assume that the following integrals exist

$$I_0^\alpha [f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \;\; t \in (0,1],$$

and

$$I_1^\alpha [f](t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} f(s) ds, \;\; t \in [0,1).$$

Then we call them the left-side, and the right-side, Riemann–Liouville integral operator of the fractional order $\alpha > 0$, respectively.

**Definition 2.2.** Define left-side and right-side Riemann–Liouville differential operators of the fractional order $\alpha$ $(0 < \alpha < 1)$ by

$$D_0^\alpha [f](t) = \frac{d}{dt} I_0^{1-\alpha} [f](t) \;\text{ and }\; D_1^\alpha [f](t) = -\frac{d}{dt} I_1^{1-\alpha} [f](t),$$

respectively.

**Definition 2.3.** For $0 < \alpha < 1$ we say that the actions

$$\mathcal{D}_0^\alpha [f](t) = D_0^\alpha [f(t) - f(0)] \;\text{ and }\; \mathcal{D}_1^\alpha [f](t) = D_1^\alpha [f(t) - f(1)],$$

are left-side and right-side differential operators of the fractional order $\alpha$ $(0 < \alpha < 1)$ in the Caputo sense, respectively.

Note that in monographs [12,9,6], authors studied different types of fractional differentiations and their properties. In what follows we formulate statements of necessary properties of integral and integro-differential operators of the Riemann–Liouville type and fractional Caputo operators.

**Property 2.4.** *[9, page 34] Let $u, v \in L^2(0,1)$ and $0 < \alpha < 1$. Then we have the formula of integration by parts*