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# On zero-sector reducing operators ☆

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ABSTRACT

We prove a Jensen-disc type theorem for polynomials $p \in \mathbb{R}[z]$ having all their zeros in a sector of the complex plane. This result is then used to prove the existence of a collection of linear operators $T: \mathbb{R}[z] \to \mathbb{R}[z]$ which map polynomials with their zeros in a closed convex sector $|\arg z| \le \theta < \pi/2$ to polynomials with zeros in a smaller sector $|\arg z| \le \gamma < \theta$. We, therefore, provide the first example of a *zero-sector reducing operator*.

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## 1. Introduction

For any polynomial $p \in \mathbb{C}[z]$, the classical Gauss–Lucas Theorem states that the zeros of the derivative $p'$ lie inside the closed convex hull of the zeros of $p$. Jensen proved a more precise result in the case where the polynomial $p$ has real coefficients. Jensen's theorem states that all of the non-real zeros of the derivative of a polynomial $p \in \mathbb{R}[x]$ must lie in at least one of the Jensen discs for $p$, where a *Jensen disc* for $p$ is a closed disc whose diameter connects a conjugate pair of non-real zeros of $p$ [4].

Either of the results just mentioned demonstrate that the differentiation operator on $\mathbb{R}[x]$ (or $\mathbb{C}[x]$, in the case of the Gauss–Lucas Theorem) maps polynomials with zeros in a strip

$$\sigma(A) = \{z \colon |\operatorname{Im} z| \le A\}$$

to polynomials with zeros in that same strip. Thus, differentiation is an example of a *zero-strip preserving operator*. Bleecker and Csordas use a result of de Bruijn to demonstrate that some differential operators

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such as $\exp(-\alpha^2 D^2/2)$, where $\alpha > 0$, map polynomials with zeros in the strip $\sigma(A)$ to a strictly smaller strip $\sigma(A')$, where $A' = \sqrt{\max\{A^2 - \alpha^2, 0\}}$ (see [1, Theorem 3.2]). Such operators are called *complex zero-strip decreasing operators* and they have been studied in detail by the first author [2].

For a closed convex sector

$$S(\theta) = \{z\colon |\arg z| \le \theta \text{ or } z = 0\} \qquad (0 \le \theta < \pi/2),$$

there are known results which demonstrate the existence of *zero-sector preserving operators* (see, for example, [3, Chapter 4]). One of the main results of this paper is to demonstrate the existence of a collection of zero-sector *reducing* operators (Theorem 2.5). We do so by proving a Jensen disc-type theorem for polynomials with their zeros in a sector (Theorem 2.2).

In the extreme case, the strip degenerates to the real line and the sector degenerates to the non-negative real axis. In [8], Pólya and Schur characterized all linear operators on $\mathbb{R}[z]$ of the form $T[z^n] = \gamma_n z^n$ which preserve the location of zeros on these limiting sets. They termed the sequence $\{\gamma_k\}_{k=0}^{\infty}$ corresponding to an operator $T$ which maps polynomials with only real zeros to polynomials with only real zeros a *multiplier sequence of the first kind*. Similarly, they termed the sequence $\{\gamma_k\}_{k=0}^{\infty}$ corresponding to an operator $T$ which maps polynomials with only positive real zeros to polynomials with only real zeros a *multiplier sequence of the second kind*. A multiplier sequence of the second kind can be thought of as an operator which maps polynomials with zeros in the sector $S(0)$ to polynomials with zeros in the double sector

$$\pm S(0) = \{z\colon z \in S(0) \text{ or } -z \in S(0)\}.$$

Our results will yield new proofs of some of the classical results. In particular, we will provide a new proof of a result due to Laguerre [5] which states that the sequence $\{\cos(\lambda + k\theta)\}_{k=0}^{\infty}$, where $\lambda$ and $\theta$ are real, is a multiplier sequence of the second kind (Corollary 2.3).

**Remark 1.** Throughout this paper, we will continue to use the notation $\sigma(A)$, $S(\theta)$, and $\pm S(\theta)$ to denote the strip, sector, and double sector, respectively, as defined in this introduction.

## 2. Some zero-sector reducing operators

We next extend the notion of a Jensen disc from the setting of horizontal strips containing the roots of real polynomials to the case in which the roots of real polynomials belong to a sector.

**Definition 2.1.** Suppose $a$ and $b$ are positive real numbers and $a+ib$ is a zero of $p \in \mathbb{R}[z]$ with $\arg(a+ib) = \theta$. For $0 \le \alpha \le \pi$ and $|\sec\alpha| < \sec\theta$, we define the **Jensen sector-disc** corresponding to $a+ib$ and $\alpha$ as the closed disc $\Delta(a, b; \alpha)$ with center $c = (\cos\alpha)(a^2+b^2)/a = a\cos\alpha\sec^2\theta$, and radius $r = \sqrt{c^2 - a^2 - b^2}$. In the case $|\sec\alpha|$ is not less than $\sec\theta$, we define $\Delta(a, b; \alpha) = \emptyset$. The Jensen sector-disc is depicted in Fig. 1.

**Remark 2.** Geometrically, the Jensen sector-disc $\Delta(a, b; \alpha)$ is the disc which is tangent to the two rays $\arg z = \pm\gamma$, where $\cos\gamma = \cos\theta\sec\alpha$, with the points of tangency lying on the circle $|z| = |a+ib|$ (see Fig. 1).

**Theorem 2.2.** *Suppose*

$$p(z) = \prod_{k=1}^{m}(z - x_k) \cdot \prod_{k=1}^{n}[(z - a_k)^2 + b_k^2],$$

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