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Riesz operators with finite rank iterates

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A B S T R A C T

Every infinite dimensional Banach space admits Riesz operators that are not finite rank. In this note we discuss conditions under which a Riesz operator, or some power thereof, is a finite rank operator.

0. Introduction

Each infinite dimensional Banach space admits Riesz operators all of whose powers are infinite rank; see for instance Proposition 2.3, or Examples 2.2 and 4.2 (including the remarks following it) and Theorem 4.5 for examples of such operators defined on particular spaces. We investigate conditions that ensure that a Riesz operator, or some power of it, is finite rank.

This line of research originates in a result of Ghahramani [6, Theorem 1], who proved that a compact homomorphism defined on a C^*-algebra is a finite rank operator. Mathieu [11] generalised this result by proving that a weakly compact homomorphism defined on a C^*-algebra with range in a normed algebra is a finite rank operator. More recently, Koumba and the second named author [7, Example 3.1] have given an example of a homomorphism defined on a C^*-algebra that is a Riesz operator, but not a finite rank operator. However, if a homomorphism T defined on a C^*-algebra is a Riesz operator with finite ascent n, then T^n is a finite rank operator [7, Theorem 3.3]. In the present work, we seek similar results beyond the class of homomorphisms, that is, we consider Riesz operators defined on general Banach spaces.

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1. Preliminaries

Let X be a Banach space. Throughout this paper, the Banach algebra of all bounded linear operators on X will be denoted by $\mathcal{L}(X)$ and the closed twosided ideal of all compact operators in $\mathcal{L}(X)$ by $\mathcal{K}(X)$. An operator $T \in \mathcal{L}(X)$ is called a *Riesz operator* if the coset $T + \mathcal{K}(X)$ is quasinilpotent in the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$. The collection of these operators will be denoted by $\mathcal{R}(X)$. An operator $T \in \mathcal{L}(X)$ is called an *inessential operator* if the coset $T + \mathcal{K}(X)$ belongs to the radical of the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Hence, every inessential operator is a Riesz operator. The collection of inessential operators on a Banach space is the largest ideal consisting of Riesz operators and this ideal will be denoted by $\mathcal{I}(X)$. An operator $T \in \mathcal{L}(X)$ is called *strictly singular* if there is no infinite dimensional closed subspace Z of X such that $T : Z \to T(Z)$, the restriction of T to Z, is an isomorphism. The closed ideal of strictly singular operators in $\mathcal{L}(X)$ will be denoted by $\mathcal{S}(X)$. An operator $T \in \mathcal{L}(X)$ is called *nuclear* if there are sequences (y_n) in X and (f_n) in X^* such that

$$\sum_{n=1}^{\infty} \|f_n\| \, \|y_n\| < \infty \qquad \text{and} \qquad Tx = \sum_{n=1}^{\infty} f_n(x) y_n \qquad (x \in X).$$

The ideal of nuclear operators in $\mathcal{L}(X)$ will be denoted by $\mathcal{N}(X)$. If $\mathcal{F}(X)$ denotes the ideal of finite rank operators on X, then it is well known that

$$\mathcal{F}(X) \subset \mathcal{N}(X) \subset \mathcal{K}(X) \subset \mathcal{S}(X) \subset \mathcal{I}(X) \subset \mathcal{R}(X). \tag{1.1}$$

Unlike the other sets in (1.1), $\mathcal{R}(X)$ is in general not an ideal. Also, there are examples to illustrate that the inclusions can be proper. However, there are Banach spaces for which some of these ideals coincide. For instance, if H is a Hilbert space then $\mathcal{K}(H) = \mathcal{S}(H) = \mathcal{I}(H)$. We refer the reader to [5] for basic properties of Riesz operators.

For $T \in \mathcal{L}(X)$, denote the *null space* of T by $N(T)$ and the *range* of T by $R(T)$. The smallest nonnegative integer n such that $N(T^n) = N(T^{n+1})$ is called the *ascent* of T and it is denoted by $\alpha(T)$. If no such n exists, set $\alpha(T) = \infty$. The *descent* of T, $\delta(T)$, is the smallest nonnegative integer n such that $R(T^n) = R(T^{n+1})$. If no such n exists, set $\delta(T) = \infty$. The *nullity* of $T \in \mathcal{L}(X)$ is $n(T) = \dim N(T)$ and the *deficiency* of $T \in \mathcal{L}(X)$ is $d(T) = \dim X/R(T)$. An operator $T \in \mathcal{L}(X)$ is called a *semi Fredholm* operator if it has closed range and it has finite nullity or finite deficiency. It is called an *upper semi Fredholm* operator if it has closed range and finite nullity. It is called a *lower semi Fredholm* operator if it has finite deficiency (in which case $R(T)$ is automatically closed).

2. Riesz operators with finite ascent

This section is motivated by the following result from [7].

Theorem 2.1. ([7], Theorem 2.2) *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be a Riesz operator with $\alpha(T) = k < \infty$. If $R(T^k) + N(T^k)$ is closed in X, then T^k is a finite rank operator.*

Our next example illustrates that there exist Riesz operators with finite ascent such that no power of T is finite rank. This shows in particular that the hypothesis that $R(T^k) + N(T^k)$ be closed cannot be omitted in Theorem 2.1. One such example is the Volterra operator.

Example 2.2. Let $X = C[0, 1]$ be the Banach space of all complex valued continuous functions defined on the interval $[0, 1]$ and let $T \in \mathcal{L}(X)$ be the Volterra operator given by

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