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Bounds for modified Struve functions of the first kind and their ratios

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ABSTRACT

We obtain a simple two-sided inequality for the ratio $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$ in terms of the ratio $I_\nu(x)/I_{\nu-1}(x)$, where $\mathbf{L}_\nu(x)$ is the modified Struve function of the first kind and $I_\nu(x)$ is the modified Bessel function of the first kind. This result allows one to use the extensive literature on bounds for $I_\nu(x)/I_{\nu-1}(x)$ to immediately deduce bounds for $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$. We note some consequences and obtain further bounds for $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$ by adapting techniques used to bound the ratio $I_\nu(x)/I_{\nu-1}(x)$. We apply these results to obtain new bounds for the condition numbers $x\mathbf{L}_\nu'(x)/\mathbf{L}_\nu(x)$, the ratio $\mathbf{L}_\nu(x)/\mathbf{L}_\nu(y)$ and the modified Struve function $\mathbf{L}_\nu(x)$ itself. Amongst other results, we obtain two-sided inequalities for $x\mathbf{L}_\nu'(x)/\mathbf{L}_\nu(x)$ and $\mathbf{L}_\nu(x)/\mathbf{L}_\nu(y)$ that are given in terms of $xI_\nu'(x)/I_\nu(x)$ and $I_\nu(x)/I_\nu(y)$, respectively, which again allows one to exploit the substantial literature on bounds for these quantities. The results obtained in this paper complement and improve existing bounds in the literature.

1. Introduction

The ratios of modified Bessel functions $I_\nu(x)/I_{\nu-1}(x)$ and $K_{\nu-1}(x)/K_\nu(x)$ arise in many areas of the applied sciences, including epidemiology [32], chemical kinetics [28] and signal processing [23]; see [39] and references therein for further applications. These ratios are also key computational tools in the construction of numerical algorithms for computing modified Bessel functions (see, for example, Algorithms 12.6 and 12.7 of [13]). There is now an extensive literature on lower and upper bounds for these ratios; see [1,5,14,15,17–19,26,27,30,31,36,38–40,43]. There is also a considerable literature on lower and upper bounds for the ratios $I_\nu(x)/I_\nu(y)$ and $K_\nu(x)/K_\nu(y)$; see [1,2,4,10,11,17,20,21,24,25,35,37,42], which has been used, for example, to obtain tight bounds for the generalized Marcum Q-function, which arises in radar signal processing [2,10].

The modified Struve functions are related to the modified Bessel functions. Likewise, they arise in manyfold applications, including leakage inductance in transformer windings [16], perturbation approximations of lee waves in a stratified flow [29], scattering of plane waves by circular cylinders [44] and lift and downwash

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distributions of oscillating wings in subsonic and supersonic flow [45,46]; see [6] for a list of further application areas.

The first detailed study of inequalities for modified Struve functions was [22], in which two-sided inequalities for modified Struve functions and their ratios were obtained, together with Turán and Wronski type inequalities. Recently, [7] used a classical result on the monotonicity of quotients of Maclaurin series and techniques developed in the extensive study of modified Bessel functions and their ratios to obtain monotonicity results and, as a consequence, functional inequalities for the modified Struve function of the first kind $\mathbf{L}_\nu(x)$ that complement and improve the results of [22]. Further results and a new proof of a Turán-type inequality for the modified Struve function of the first kind are given in [9], and monotonicity results and functional inequalities for the modified Struve function of the second kind $\mathbf{M}_\nu(x) = \mathbf{L}_\nu(x) - I_\nu(x)$ are given in [8]. It should be noted that the techniques used in [7] and [8] to obtain functional inequalities for $\mathbf{L}_\nu(x)$ and $\mathbf{M}_\nu(x)$ are quite different (this is also commented on in [12]), which is in contrast to the literature on modified Bessel functions in which functional inequalities for $I_\nu(x)$ and $K_\nu(x)$ are often developed in parallel. For this reason, in this paper, with the exception of Remark 2.5, we restrict our attention to the modified Struve function $\mathbf{L}_\nu(x)$.

In this paper, we obtain new bounds for the ratios $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$ and $\mathbf{L}_\nu(x)/\mathbf{L}_\nu(y)$, the condition numbers $x\mathbf{L}_\nu'(x)/\mathbf{L}_\nu(x)$ and the modified Struve function $\mathbf{L}_\nu(x)$ itself. These results complement and, at least in some cases, improve those given in [7,22]. Our approach is quite different, though. In Section 2, we obtain a simple but accurate two-sided inequality for the ratio $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$ in terms of the ratio $I_\nu(x)/I_{\nu-1}(x)$. This result is quite powerful because it allows one to exploit the extensive literature on bounds for $I_\nu(x)/I_{\nu-1}(x)$ to bound $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$. We give some examples, and complement these bounds by showing that some of the techniques from the literature used to bound $I_\nu(x)/I_{\nu-1}(x)$ can be easily adapted to bound the ratio $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$. In Section 3, we apply these bounds to obtain new bounds for the quantities $x\mathbf{L}_\nu'(x)/\mathbf{L}_\nu(x)$, $\mathbf{L}_\nu(x)/\mathbf{L}_\nu(y)$ and the modified Struve function $\mathbf{L}_\nu(x)$. Amongst other results, we obtain two-sided inequalities for $x\mathbf{L}_\nu'(x)/\mathbf{L}_\nu(x)$ and $\mathbf{L}_\nu(x)/\mathbf{L}_\nu(y)$ that are given in terms of $xI_\nu'(x)/I_\nu(x)$ and $I_\nu(x)/I_\nu(y)$, respectively, which again allows one to exploit the substantial literature on these quantities. Through a combination of asymptotic analysis of the bounds and numerical results, we find that, in spite of their simple form, the bounds obtained in this paper are quite accurate and often tight in certain limits.

2. Upper and lower bounds for the ratio $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$

2.1. *Bounding* $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$ *via bounds for* $I_\nu(x)/I_{\nu-1}(x)$

In this section, we obtain a simple but accurate double inequality for the ratio $\mathbf{L}_\nu(x)/\mathbf{L}_{\nu-1}(x)$ in terms of the ratio $I_\nu(x)/I_{\nu-1}(x)$. The modified Bessel and modified Struve functions $I_\nu(x)$ and $\mathbf{L}_\nu(x)$ are closely related functions that have the following power series representations (see [33] for these and the forthcoming properties):

$$I_\nu(x) = \sum_{n=0}^\infty \frac{(\frac{1}{2}x)^{2n+\nu}}{n!\Gamma(n+\nu+1)}, \tag{2.1}$$

$$\mathbf{L}_\nu(x) = \sum_{n=0}^\infty \frac{(\frac{1}{2}x)^{2n+\nu+1}}{\Gamma(n+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})}. \tag{2.2}$$

It is immediate from these series representations, and the standard formulas $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ and $t\Gamma(t) = \Gamma(t+1)$ that, as $x \downarrow 0$,

$$I_\nu(x) \sim \frac{x^\nu}{2^\nu\Gamma(\nu+1)}, \quad \nu > -1, \tag{2.3}$$