ARTICLE IN PRESS
APAL:2650

Annals of Pure and Applied Logic ••• (••••) •••–•••

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Algorithmically random series and Brownian motion

Paul Potgieter

Department of Decision Sciences, University of South Africa, P.O. Box 392, Pretoria 0003, South Africa

ARTICLE INFO

Article history:
Received 3 April 2017
Received in revised form 15 June 2018
Accepted 28 June 2018
Available online xxxx

MSC:
68Q30
42A20
42A38
60G15

Keywords:
Martin-Löf randomness
Rademacher series
Fourier series
Brownian motion

ABSTRACT

We consider some random series parametrised by Martin-Löf random sequences. The simplest case is that of Rademacher series, independent of a time parameter. This is then extended to the case of Fourier series on the circle with Rademacher coefficients. Finally, a specific Fourier series which has coefficients determined by a computable function is shown to converge to an algorithmically random Brownian motion.

1. Introduction

Since infinite series provide a natural model for games of chance, a recurring problem in early twentieth-century probability theory was the summation of series with random coefficients. There was, however, no satisfying notion of what constitutes a random sequence. The attempt to formalise the notion informed the development of the field, and for instance served as the basis of von Mises' axiomatisation of probability theory [17]. This was of course supplanted by Kolmogorov's axiomatisation [10], and the problem of characterising randomness did not remain central to probability theory.

The work of Borel [2] can be considered the first use of the concept of randomness in analysis, according to [14], and concerns averages of sums of randomly determined quantities. A similar type of problem to that which concerned Borel can be phrased, in modern terms, as a problem of *Rademacher series*. This can be seen as a model of an infinite coin toss, with different wagers being placed on each successive toss.

E-mail address: potgip@unisa.ac.za.

https://doi.org/10.1016/j.apal.2018.07.001

A Rademacher sequence on a probability space $(\Omega, \Sigma, \mathbb{P})$ is defined as a sequence (ε_n) of random variables on Ω such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$, $n = 1, 2, \ldots$. A real Rademacher series is one of the form

$$\sum_{n=1}^{\infty} \varepsilon_n u_n, \tag{1.1}$$

where (ε_n) is a Rademacher sequence and (u_n) is a sequence of given real numbers. We shall focus on Rademacher sequences generated by the binary expansion of reals in $[0, 1]$, with $\varepsilon_n(x) = 1$ if the nth digit in the expansion of x is 1, and $\varepsilon_n(x) = -1$ otherwise. Steinhaus [16] first showed that such series converge almost surely with respect to Lebesgue measure when $\sum u_n^2$ converges, and diverge a.s. otherwise.

Major advances in the early study of random series were published by Paley and Zygmund in 1930 [15], and later in collaboration with Wiener [14]. Most relevant to our purposes was their study of the convergence or divergence of series of the form

$$\sum_{n=-\infty}^{\infty} \varepsilon_n a_n e^{int}, \tag{1.2}$$

where (ε_n) is a Rademacher sequence as above and the (a_n) are fixed reals. Similar results to those of Steinhaus were obtained, in that almost sure (with respect to Lebesgue measure) convergence or divergence of such series were dictated by the convergence or divergence of the series $\sum a_n^2$.

One might ask whether the convergence or divergence of the above series can contribute to the discussion of what constitutes a random sequence or number. Of course, one could reasonably expect reals in the interval $[0, 1]$ to be almost surely random. Indeed, Borel already recognised that there can be only countably many elements of the 'practical' continuum, consisting of reals determined by some law [18], which therefore has measure 0. However, the null sets in the above series where convergence fails are not uniquely specified, and we cannot classify numbers as random or not based solely on measure theoretic terms.

Several notions of randomness arose from the theory of computability, some of which were fittingly informed by measure theoretic concepts in the form of "tests" for randomness (see, for instance, [13] for an overview). The purpose of this paper is to explore whether it is possible to reconcile one of the formulations of algorithmic randomness with the idea of randomness dictated by the convergence or divergence of certain series. Our results show that, with certain computability restrictions, Martin-Löf randomness provides such a formulation.

The connection between Martin-Löf randomness and almost sure results in probability is certainly not new. Most germane to the final part of this paper is the construction of algorithmically random Brownian motion by Fouché. In [7], it is shown that there is a bijection between the set of Martin-Löf random sequences and a subset of $C[0, 1]$ of Wiener measure 1, the so-called complex oscillations, which reflect many properties of Brownian motion, such as the law of the iterated logarithm [9]. This bijection was achieved through the use of Franklin–Wiener series. Using the results in this paper, we show that complex oscillations may also be constructed through the use of Fourier–Wiener series, similar to Wiener's original construction of Brownian motion in [19]. In this paper we shall use Wiener's construction as presented in the book of Kahane [8].

1.1. Structure of the paper

Section 2.1 presents the necessary background in algorithmic randomness that will be used throughout the paper. In Section 2.2, we present some definitions that will be needed, as well as the main theorems regarding random series that we will find corresponding algorithmically random versions of in Sections 3 to 5. Section 2.3 provides all of the necessary background on algorithmically random Brownian motion required for Section 6, but is not required for earlier sections of the paper.

Please cite this article in press as: P. Potgieter, Algorithmically random series and Brownian motion, Ann. Pure Appl. Logic (2018), https://doi.org/10.1016/j.apal.2018.07.001