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The homotopy theory of type theories

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ABSTRACT

We construct a left semi-model structure on the category of intensional type theories (precisely, on $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$). This presents an ∞-category of such type theories; we show moreover that there is an ∞-functor Cl_∞ from there to the ∞-category of suitably structured quasi-categories.
This allows a precise formulation of the conjectures that intensional type theory gives internal languages for higher categories, and provides a framework and toolbox for further progress on these conjectures.

1. Introduction

Homotopy Type Theory (HoTT) arises from the discovery that the logical system of dependent type theory can be naturally interpreted in homotopy-theoretic settings, and provides a rich language for such settings, thanks largely to its richer treatment of equality compared to first-order logic.

Classically, equality carries no information beyond its truth value; but in type theory, objects can be equal/identifiable in a variety of ways, so a type of equalities may be a

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non-trivial type in its own right, analogous to the path space of a topological space. This forms the basis of "synthetic homotopy theory", a major direction within HoTT: the development of homotopy-theoretic constructions and theorems, entirely elementarily within type theory.

Of course, one wants to know that these can be interpreted in a good range of established settings that a homotopy theorist might care about. Conversely, one may hope that all homotopy-theoretic concepts (in some sense) can be translated into type theory. Various results in these directions have been given — some proven, some conjectured, some only informally sketched (see e.g. [14,24–27,17]).

Such hopes have been summarized as the idea that HoTT should be the **internal language of ∞-categories**. Precisely, by analogy with established "internal languages" in settings such as topos theory, this should mean a single master statement subsuming the above results: the existence of a suitable equivalence between some (higher) categories of type theories and ∞-categories.

The first contribution of the present paper is a framework for formulating such a claim precisely. We do so by assembling type theories into a higher category, and giving a functor Cl_∞ from this to a higher category of suitably structured quasicategories. The internal language conjecture can then be stated as: Cl_∞ is an equivalence of higher categories.

The other main contribution is a left semi-model structure on the category of type theories. This gives a tractable and explicit presentation of the higher category thereof, which we hope will provide a solid base for further progress on the conjectures.

In a little more detail: we work with "type theories" as contextual categories or categories with attributes (CwA's), keeping our results independent of the correspondence between these and syntactically presented theories. We assume Id-, Σ-, and unit types throughout; we consider also the extension to Π-types.

Two technical tools of the paper may be of independent interest. One (small, but useful and to our knowledge new) is a notion of equivalence between arbitrary objects of a CwA. The other is the construction of the CwA of span-equivalences in a given CwA, a powerful tool for constructing equivalences between CwA's.

In Section 5, we make use of some results from our forthcoming article [19], currently in preparation. However, those results may be treated as black boxes; the present paper can be read as essentially self-contained.

During the preparation of this paper, we learned that Valery Isaev has independently given a similar construction in [12], defining a (full) model structure on a slightly different category of type theories (assuming an interval type, instead of Martin-Löf identity types).

2. Background

In this section, we review the necessary background on categorical models of type theory. We recall the definition of a contextual category and introduce the notation for