Advances in Mathematics 337 (2018) 39–82

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Manin's conjecture for certain spherical threefolds

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A R T I C L E   I N F O

Article history:
Received 15 November 2016
Received in revised form 16 July 2018
Accepted 1 August 2018
Available online xxxx
Communicated by Ravi Vakil

MSC:
primary 11D45
secondary 14M27, 14G05, 11G35

Keywords:
Rational points
Manin's conjecture
Spherical varieties
Fano threefolds

A B S T R A C T

We prove Manin's conjecture on the asymptotic behavior of the number of rational points of bounded anticanonical height for a spherical threefold with canonical singularities and two infinite families of spherical threefolds with log terminal singularities. Moreover, we show that one of these families does not satisfy a conjecture of Batyrev and Tschinkel on the leading constant in the asymptotic formula. Our proofs are based on the universal torsor method, using Brion's description of Cox rings of spherical varieties.

Contents

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https://doi.org/10.1016/j.aim.2018.08.005

1. Introduction

1.1. Spherical varieties and Manin's conjecture

Manin's conjecture [28,4,39,6,40,41] makes a precise prediction for the asymptotic behavior of the number of rational points of bounded anticanonical height on (almost) Fano varieties over number fields whose set of rational points is Zariski dense.

For a smooth Fano variety over $\mathbb{Q}$ with a Zariski dense set of rational points, one may introduce an anticanonical height function $H\colon X(\mathbb{Q}) \to \mathbb{R}_{>0}$ and ask for the asymptotic behavior of the number of rational points of bounded height, as the height bound tends to infinity. The total number might be dominated by points on *accumulating* subvarieties (or, more generally, accumulating *thin subsets*, see [40, §8]), and hence it is more interesting to restrict to their complement U. By [4, Conjecture B'], we are led to the expectation that

$$N_{U,H}(B) \coloneqq \#\{x \in U(\mathbb{Q})\colon H(x) \leq B\} \sim \mathfrak{c}B(\log B)^{\rho-1}$$

as $B \to \infty$, where ρ is the Picard number of X. A conjecture for the leading constant $\mathfrak{c}$ is given by Peyre in [39]. If X is a singular Fano variety with a *crepant resolution* $\pi\colon \widetilde{X} \to X$ (i.e., a desingularization with $\pi^*(-K_X) = -K_{\widetilde{X}}$), then [4, Conjecture C'] and [40, 5.1] tell us that such an asymptotic formula should hold with ρ and $\mathfrak{c}$ computed on $\widetilde{X}$. If X has worse singularities, [4, Conjecture C'] and [40, 3.6] predict

$$N_{U,H}(B) \sim \mathfrak{c}B^{\mathfrak{a}}(\log B)^{\mathfrak{b}-1},$$

where we may have $\mathfrak{a} > 1$; Batyrev and Tschinkel [6] give a prediction for $\mathfrak{c}$.

Manin's conjecture has been proved for some classes of varieties and several individual examples. Most of the known cases are proved using either harmonic analysis on adelic points or the universal torsor method combined with various analytic techniques.

Many of them are *spherical varieties*, i.e., normal G-varieties containing a dense B-orbit, where G is a connected reductive group and $B \subseteq G$ is a Borel subgroup. Spherical varieties are a huge class of varieties that admit a combinatorial description by spherical systems (Luna's program [36]) and colored fans (Luna–Vust theory [37]) generalizing the combinatorial description of toric varieties.

In particular, harmonic analysis has been used to prove Manin's conjecture for some classes of equivariant compactifications of algebraic groups, for example flag varieties [28],