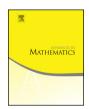


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# Boundary operators associated to the $\sigma_k$ -curvature



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#### ABSTRACT

We study conformal deformation problems on manifolds with boundary which include prescribing  $\sigma_k \equiv 0$  in the interior. In particular, we prove a Dirichlet principle when the induced metric on the boundary is fixed and an Obata-type theorem on the upper hemisphere. We introduce some conformally covariant multilinear operators as a key technical tool.

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#### 1. Introduction

Let  $(X^{n+1}, g_0)$  be a compact Riemannian manifold. The Ricci decomposition

$$Rm = W + P \wedge g_0$$

of the Riemann curvature tensor Rm into the conformally invariant Weyl curvature W and the Kulkarni–Nomizu product of the Schouten tensor

$$P := \frac{1}{n-1} \left( \text{Ric} - \frac{R}{2n} g_0 \right)$$

and the metric  $g_0$  implies that the behavior of the full Riemann curvature tensor under conformal deformation is completely controlled by the Schouten tensor. For this reason, Viaclovsky initiated [13] the study of the conformal properties of the  $\sigma_k$ -curvatures  $\sigma_k^g$ ; i.e.  $\sigma_k^g := \sigma_k(g^{-1}P_g)$  is the k-th elementary symmetric functions of the Schouten tensor of g. Note that  $\sigma_1^g = R_g/2n$  is proportional to the scalar curvature of g. Crucially, the  $\sigma_k$ -curvatures are variational if and only if  $k \leq 2$  or  $g_0$  is locally conformally flat [1, 13]. In particular, if  $k \leq 2$  or  $g_0$  is locally conformally flat, then the total  $\sigma_k$ -curvature functional,  $\mathcal{F}(g) := \int_X \sigma_k^g \, \mathrm{dvol}_g$ , is such that

$$\frac{d}{dt}\Big|_{t=0} \mathcal{F}_k\left(e^{2t\Upsilon}g\right) = (n+1-2k) \int_{\mathcal{X}} \Upsilon \sigma_k^g \, \mathrm{dvol}_g \tag{1.1}$$

for all metrics g in the conformal class  $[g_0]$  and all  $\Upsilon \in C_0^{\infty}(X)$ . Equation (1.1) realizes  $\sigma_k$  as the conformal gradient of  $\mathcal{F}_k$  when  $n+1 \neq 2k$ ; Brendle and Viaclovsky found [2] a different functional with conformal gradient  $\sigma_k$  when n+1=2k.

When  $X^{n+1}$  has nonempty boundary  $M^n$ , S. Chen introduced [6] the  $H_k$ -curvatures as a family of invariants on M which are polynomial in the restriction  $P|_{TM}$  of the Schouten tensor to M and the second fundamental form A of M. For example,  $H_1 = \frac{1}{n} \operatorname{tr} A$  is the mean curvature; see Section 2 for the general formula for  $H_k$ . A key property of the  $H_k$ -curvatures is that they enable the study of conformal deformations of the  $\sigma_k$ -curvature on manifolds with boundary by variational methods: If  $k \leq 2$  or  $g_0$  is locally conformally flat, then the functional

$$S_k(g) := \int_X \sigma_k^g \operatorname{dvol}_g + \oint_M H_k^g \operatorname{dvol}_{i^*g},$$

where  $i: M \to X$  is the inclusion mapping, satisfies

$$\frac{d}{dt}\Big|_{t=0} \mathcal{S}_k\left(e^{2t\Upsilon}g\right) = (n+1-2k) \left| \int\limits_X \Upsilon \sigma_k^g \operatorname{dvol}_g + \oint\limits_M \Upsilon H_k^g \operatorname{dvol}_{i^*g} \right|$$
(1.2)

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