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Boundary operators associated to the σ_k-curvature

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A B S T R A C T

We study conformal deformation problems on manifolds with boundary which include prescribing $\sigma_k \equiv 0$ in the interior. In particular, we prove a Dirichlet principle when the induced metric on the boundary is fixed and an Obata-type theorem on the upper hemisphere. We introduce some conformally covariant multilinear operators as a key technical tool.

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1. Introduction

Let (X^{n+1}, g_0) be a compact Riemannian manifold. The Ricci decomposition

$$\mathrm{Rm} = W + P \wedge g_0$$

of the Riemann curvature tensor Rm into the conformally invariant Weyl curvature W and the Kulkarni–Nomizu product of the Schouten tensor

$$P := \frac{1}{n-1}\left(\mathrm{Ric} - \frac{R}{2n} g_0\right)$$

and the metric g_0 implies that the behavior of the full Riemann curvature tensor under conformal deformation is completely controlled by the Schouten tensor. For this reason, Viaclovsky initiated [13] the study of the conformal properties of the σ_k-*curvatures* σ_k^g; i.e. $\sigma_k^g := \sigma_k(g^{-1}P_g)$ is the k-th elementary symmetric functions of the Schouten tensor of g. Note that $\sigma_1^g = R_g/2n$ is proportional to the scalar curvature of g. Crucially, the σ_k-curvatures are variational if and only if $k \leq 2$ or g_0 is locally conformally flat [1, 13]. In particular, if $k \leq 2$ or g_0 is locally conformally flat, then the total σ_k-curvature functional, $\mathcal{F}(g) := \int_X \sigma_k^g \,\mathrm{dvol}_g$, is such that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_k\left(e^{2t\Upsilon} g\right) = (n+1-2k) \int_X \Upsilon \sigma_k^g \,\mathrm{dvol}_g \tag{1.1}$$

for all metrics g in the conformal class $[g_0]$ and all $\Upsilon \in C_0^\infty(X)$. Equation (1.1) realizes σ_k as the conformal gradient of $\mathcal{F}_k$ when $n+1 \neq 2k$; Brendle and Viaclovsky found [2] a different functional with conformal gradient σ_k when $n+1 = 2k$.

When X^{n+1} has nonempty boundary M^n, S. Chen introduced [6] the H_k-curvatures as a family of invariants on M which are polynomial in the restriction $P|_{TM}$ of the Schouten tensor to M and the second fundamental form A of M. For example, $H_1 = \frac{1}{n}\operatorname{tr} A$ is the mean curvature; see Section 2 for the general formula for H_k. A key property of the H_k-curvatures is that they enable the study of conformal deformations of the σ_k-curvature on manifolds with boundary by variational methods: If $k \leq 2$ or g_0 is locally conformally flat, then the functional

$$\mathcal{S}_k(g) := \int_X \sigma_k^g \,\mathrm{dvol}_g + \oint_M H_k^g \,\mathrm{dvol}_{i^*g},$$

where $i\colon M \to X$ is the inclusion mapping, satisfies

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{S}_k\left(e^{2t\Upsilon} g\right) = (n+1-2k)\left[\int_X \Upsilon \sigma_k^g \,\mathrm{dvol}_g + \oint_M \Upsilon H_k^g \,\mathrm{dvol}_{i^*g}\right] \tag{1.2}$$