IFAC Journal of Systems and Control 5 (2018) 22–29

Contents lists available at ScienceDirect

IFAC Journal of Systems and Control

journal homepage: http://www.elsevier.com/locate/ifacsc

Characterization and control of conservative and non-conservative network dynamics☆

Matthias Wildemeersch[1,*,a], Wai Hong Ronald Chan[1,b], Elena Rovenskaya[a,c], Tony Q.S. Quek[d]

[a] IIASA, Schlossplatz 1, A-2361 Laxenburg, Austria
[b] Stanford University, 440 Escondido Mall, CA 94305-3030, USA
[c] Lomonosov Moscow State University, Russia, 119991, Moscow, GSP-1, 1-52, Leninskiye Gory
[d] SUTD, 8 Somapah Road, Singapore 487372

ARTICLE INFO

Keywords:
Multi-agent networks
Diffusion process
Directed graphs
Graph Laplacian
Network control
Reachable set

ABSTRACT

Diffusion processes are instrumental to describe the movement of a continuous quantity in a network of interacting agents. Here, we present a framework for diffusion in networks and study in particular two classes of agent interactions depending on whether the total network quantity follows a conservation law. Focusing on probabilistic, asymmetric interactions between agents, we define how the dynamics of conservative and non-conservative networks relate to the weighted in-degree and out-degree Laplacians. For uncontrolled networks, we compare the convergence behavior of both types of networks as a function of the eigenvectors of the weighted graph Laplacians. For networks with exogenous controls, we also analyze convergence and provide a method to measure the difference between conservative and non-conservative network dynamics based on the comparison of their respective reachable sets. The presented network control framework enables the comparative study of the dynamic and asymptotic network behavior for conservative and non-conservative networks.

1. Introduction

Multi-agent network dynamics received ample research interest over the last decade in the context of group coordination (Jadbabaie, Lin, & Morse, 2003) distributed algorithms (Dimakis, Kar, Moura, Rabbat, & Scaglione, 2010), network control (Pasqualetti, Zampieri, & Bullo, 2014), distributed optimization (Chen & Sayed, 2013), consensus problems (Acemoglu & Ozdaglar, 2011; Erkan & Akar, 2016; Mallada, Freeman et al., 2016; Proskurnikov, 2011; Wang & Hong, 2008), and herding and flocking behavior (Blondel, Hendrickx, Olshevsky, & Tsitsiklis, 2005). Network dynamics involve inter-agent interactions that lead to the diffusion of a continuous quantity within a network (Acemoglu, Como, Fagnani, & Ozdaglar, 2013; Como, Savla, Acemoglu, Dahleh, & Frazzoli, 2011; Yildiz, Scaglione, & Ozdaglar, 2010). In this work, we establish a probabilistic diffusion framework that describes in continuous time the movement of such a continuous quantity within a network, accounting for the stochasticity and the nature of the interactions between agents. Going beyond symmetric, unweighted graphs (Banerjee & Jost, 2008), we focus on weighted digraphs with asymmetric update rules. The main contribution of this framework is two-fold. First, we make a connection between two linear update protocols and their corresponding network dynamics. Although these protocols are different in nature with regard to the conservation of total network quantity, they can result in identical network behavior under certain circumstances. Second, we enable the comparison between the protocols by studying the steady-state and transient behavior under both protocols, in the presence and absence of external control.

Many dynamical processes that occur over networks rely on pairwise interactions between network nodes that adjust their values according to a rule of interaction. There exists a large body of literature where the network dynamics are based on different graph Laplacian matrices (Yan, Teng, Lerman, & Ghosh, 2016). A common feature of this literature is that the total amount of quantity present in the network does not follow a conservation law (Acemoglu, Nedic, & Ozdaglar, 2008; Jadbabaie et al., 2003; Olfati-Saber, Fax, & Murray, 2007; Olfati-Saber & Murray, 2004; Tcha & Pliska, 1977). In this work instead, we

☆ The material in this paper has been presented in part at the 2015 IEEE Conference on Decision and Control (Chan, Wildemeersch, and Quek, 2015).

* Corresponding author.
E-mail addresses: wildemee@iiasa.ac.at (M. Wildemeersch), whrchan@stanford.edu (W.H.R. Chan), rovenska@iiasa.ac.at (E. Rovenskaya), tonyquek@sutd.edu.sg (T.Q.S. Quek).

[1] The first two authors have equal contributions to this research.

https://doi.org/10.1016/j.ifacsc.2018.07.002
Received 5 March 2018; Accepted 30 July 2018
Available online 04 August 2018

present an inter-agent update rule that follows a conservation law and contrast it with a non-conservative update rule that is typically found in the literature. Considering both protocols, our framework can capture a wide range of network phenomena: financial and trade assets, biochemical systems, as well as human migration represent conservative flows (Barzel & Barabási, 2013; Mirzaev & Gunawardena, 2013), while the propagation of opinions follow non-conservative network dynamics. We derive the corresponding matrix differential equation that describes the diffusion of the considered quantity over the network, and highlight the differences in transient and stationary behavior for both update rules, taking into account the effects of network asymmetry.

Furthermore, we address network control by extending the homogeneous differential equation that describes the diffusion process to its inhomogeneous form. By doing so, we can model the addition of the considered quantity to the multi-agent network. We illustrate how constant control actions can result in changes of the system matrix that governs the network dynamics, and we define the convergence behavior of networks with exogenous excitation under given constraints on the input vector and network topology. When the control actions belong to a function space, we also provide a method to define the set of reachable network states based on the support function of non-empty closed convex sets. This method allows us to analyze how the conservative and non-conservative update protocol result in different reachable sets using the Hausdorff distance between these sets.

The remainder of the paper is structured as follows. Section 2 introduces the notation used throughout the paper and the Laplacian matrices that will be instrumental to characterize the network dynamics. Section 3 introduces two essential update rules to model diffusion in networks. Section 4 discusses the stability and convergence characteristics of conservative and non-conservative networks without control inputs, and Section 5 covers network control by extending the homogeneous equations to their inhomogeneous forms. Section 5 also presents a method to calculate the reachable set of network states. Finally, Section 6 provides some concluding remarks.

2. Preliminaries

We consider a population $\mathcal{V}$ of interacting agents $\mathcal{V}_i$, where $i \in \mathcal{I} = \{1, 2, \ldots, n: n \in \mathbb{Z}^+\}$. All agents possess a continuous node property $S_i(t) \in \mathbb{R}$, $t \geq t_0$, and the node properties are gathered in the state vector $S(t) = [S_1(t) \ldots S_n(t)]^{\mathrm{T}}$, $S(t) \in \mathbb{R}^n$. Assuming that $t_0 = 0$ and given the initial conditions $S(0) = S_0$, the node properties evolve over time according to a stochastic update process where property updates occur at times determined by a Poisson process. The probabilistic interactions between the agents can be described by a weighted digraph $G = (\mathcal{V}, \vec{\mathcal{E}}, w)$, where $\mathcal{V}$ is the set of agents and $\vec{\mathcal{E}}$ is the set of directed links (i, j) between pairs of agents from $\mathcal{V}$. In this work, we consider $\mathcal{V}$ and $\vec{\mathcal{E}}$ to be constant over time. The weight function $w: \vec{\mathcal{E}} \mapsto \mathbb{R}^+$ captures for each edge in the network the update rate as well as the liability between the interacting agents. The weighted adjacency matrix can be represented as

$$A_G(i, j) = \begin{cases} w(i, j) & \text{if } (i, j) \in \vec{\mathcal{E}}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{1}$$

The weighted in-degree and out-degree matrices are diagonal matrices with diagonal elements given by

$$D_G^{(\text{in})}(j, j) = \sum_i A_G(i, j)\,, \tag{2}$$

$$D_G^{(\text{out})}(i, i) = \sum_j A_G(i, j)\,. \tag{3}$$

Since the interactions between agents can be asymmetric, we define two Laplacians that refer to the in-degree and the out-degree of each node. We define the weighted in-degree and out-degree Laplacians as

$$L_G^{(\text{in})} = D_G^{(\text{in})} - A_G\,, \tag{4}$$

$$L_G^{(\text{out})} = D_G^{(\text{out})} - A_G\,. \tag{5}$$

3. Stochastic update rules

Depending on the update rule applied in the probabilistic interactions between nodes, we characterize the flow dynamics of networks operating under different protocols. Here, we describe two main classes of linear update rules that result in linear, time-invariant matrix differential equations in the node property. These update rules are distinguished by the conservation or the non-conservation of the total property initially present in the network. The networks applying the conservative and non-conservative protocols will be referred to as conservative and non-conservative networks, respectively.

3.1. Conservative networks

We first consider a protocol where the total property in the network is conserved, i.e., $S_{tot} = S_1(t) + \cdots + S_n(t)$ is constant over time. Conservative updating is relevant for the description of conservative flow dynamics between network nodes, including the flow of material and physical assets. In this respect, conservative networks are able to represent stylized instances of hydraulic networks, human mobility (Barzel & Barabási, 2013), or biochemical systems (Mirzaev & Gunawardena, 2013). Here, agents obey the conservative update rule

$$\begin{aligned} S_i(t + \Delta t) &= S_i(t) + C_{ij} S_j(t) \\ S_j(t + \Delta t) &= (1 - C_{ij}) S_j(t)\,, \end{aligned} \tag{P1}$$

where $i, j \in \mathcal{I}$, $(i, j) \in \vec{\mathcal{E}}$. The parameter $C_{ij} \in (0, 1]$ is a measure of liability or responsibility of agent j towards agent i, and Δt is an infinitesimal time interval. On every edge $(i, j) \in \vec{\mathcal{E}}$, a stochastic process takes place on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with sample space $\Omega = \mathbb{R}_+$, the σ-algebra $\mathcal{F}$ of Borel sets on $\mathbb{R}_+$, and probability measure $\mathbb{P}$. We consider a counting process on the positive reals according to an independent, stationary Poisson process with rate $r_{ij} > 0$. The counting process has also an interpretation as a ticking clock with exponentially distributed inter-event times. The protocol (P1) is executed for nodes i and j when the independent Poisson clock of (i, j) ticks at time t. The following Lemma characterizes the property dynamics of the expected value of S in conservative networks.

Lemma 1. *Let $\bar{S}(t)$ denote the expected value of $S(t)$. The dynamics of the expected value for a network applying (P1) are defined by the governing equation*

$$\dot{\bar{S}}(t) = Q \bar{S}(t)\,, \tag{A}$$

where $Q = -L_G^{(\text{in})}$, the weight function is defined as $w(i, j) = C_{ij} r_{ij}$, and

$$Q_{ij} = \begin{cases} C_{ij} r_{ij} & \text{if } i \neq j\,, \\ -\sum_{k \neq i} C_{ki} r_{ki} & \text{if } i = j\,. \end{cases} \tag{6}$$

Proof. We first note that the total update rate for a node $i \in \mathcal{V}$ is given by $r_i = \sum_j r_{ij}$, and that the total update rate of the network is given by $r = \sum_i r_i$. Assume that a global network clock is ticking at rate r. Then, the probability that the clock will activate edge (i, j) is given by r_{ij}/r, where in the limit of large-scale networks $r \approx 1/\Delta t$. Consequently, when an outgoing edge (i, j) of node i is triggered with probability $r_{ij} \Delta t$, the probabilistic update of the node properties involved in the update according to (P1) is given by

$$\bar{S}_i(t + \Delta t) - \bar{S}_i(t) = \sum_{j \neq i} r_{ij} \Delta t\, C_{ij}\, \bar{S}_j(t) \tag{7}$$

$$\bar{S}_j(t + \Delta t) - \bar{S}_j(t) = -\sum_{j \neq i} r_{ij} \Delta t\, C_{ij}\, \bar{S}_j(t)\,. \tag{8}$$