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Polynomial time algorithm for computing a minimum geodetic set in outerplanar graphs

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ABSTRACT

Given a graph G and a pair of vertices u, v the interval $I_G[u, v]$ is the set of all vertices that are in some shortest path between u and v. Given a subset X of vertices of G, the interval $I_G[X]$ of X, is the union of the intervals for all pairs of vertices in X and we say that X is *geodetic* if its interval do coincide with the set of vertices in the graph. A minimum geodetic set is a minimum cardinality geodetic set of G. The problem of computing a minimum geodetic set is known to be NP-Hard for general graphs but is known to be polynomially solvable for maximal outerplanar graphs. In this paper we show a polynomial time algorithm for finding a minimum geodetic set in general outerplanar graphs.

1. Introduction

Given a finite set V and a family $\mathcal{L}$ of subsets of V, $\mathcal{L}$ is called *alignment* of V if and only if $\mathcal{L}$ is closed under intersection and contains both V and the empty set. Elements of $\mathcal{L}$ will be considered as *convex sets*. An *aligned space* (or a *convex space*) is a pair $(V; \mathcal{L})$, where $\mathcal{L}$ is an alignment of V. A *geodesic* is a shortest path between a pair of vertices of a graph G. A path is *chordless* if it has no chord, that is no edge of the graph joins two non consecutive vertices of the path. The *interval* of a pair of vertices u and v of G, denoted by $I_G[u, v]$, is the set of all vertices that lie on some geodesic between u and v. The interval of a set of vertices S, $I_G[S]$ is the union of the intervals between pairs of vertices of S, taken over all pairs of vertices in S.

It is immediate to see that the collection $\mathcal{L}$ of the subsets S for which $S = I_G[S]$ is an alignment on the vertex set of a graph.

Several types of convexity in graphs and hypergraphs have been defined, based on different notions of path (geodesic, monophonic and simple-path convexities [1–5]) or on properties of minimal separators (canonical convexity [6]).

However the geodesic and monophonic convexities are the most widely studied [1,7–9]. In this paper we only consider geodesic convexity and we simply refer to it as a convexity.

The *convex hull* of a set S of vertices of G, denoted as $CH_G(S)$, is the smallest convex set containing S.

A set S is a *geodetic set* of G if $I_G[S] = V(G)$. The *geodetic number* of a graph is the cardinality of the minimum set S such that $I_G[S] = V(G)$ and given a graph G, the problem of deciding if there exists a geodesic set of cardinality less than an integer $k > 1$ is the *geodetic number problem* GNP. In [10] it has been proved that the GNP is NP-complete for general graphs.

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In [1] the GNP is solved for ptolemaic graphs. In [11] it has been proved that the GNP is NP-complete for chordal or chordal bipartite graphs and is polynomially solvable for cographs and split graphs. In [12] is proved that the GNP is NP-complete even for cobipartite graphs; furthermore a block decomposition approach, to solve the GNP is given and it is used to prove that the GNP is polynomially solvable in cactus graphs. Bounds on the geodetic number are given in [11] for triangle free graphs and for unit interval graphs.

A set S is a *hull set* of G if $CH_G(S) = V(G)$. The cardinality of the minimum hull set of G is the *hull number* of G. Given a graph G, deciding whether the hull number of G is less than k is NP-complete [13]. However, in [14] this problem has been solved for several classes of graphs.

The hull number is NP-hard for bipartite graphs [14], partial cubes [15], P_9-free graphs [16] and for chordal graphs [17] but it can be computed in polynomial time for cographs [13], $(q, q-4)$-graphs [14], $\{paw, P_5\}$-free graphs [18,16], for distance-hereditary graphs [19]. A fixed parameter tractable algorithm to compute the hull number of any graph G, where the parameter can be the size of a vertex cover of G or, more generally, its neighborhood diversity is given in [18]. Bounds on the hull number are given in [14,20,21].

The problem of computing the geodetic number and the hull number has also been solved for the class of maximal outerplanar graphs [22]. In this paper, we extend the results [22] by giving a polynomial time algorithm for computing a minimum geodetic set of general outerplanar graphs. It is proved, in [22], that for every subset X of vertices in a maximal outerplanar graph, we have that $I_G[X] = I_G[I_G[X]]$. As stated in [22] (page 79), this result is also valid for general biconnected outerplanar graphs. Based on this, we have that the hull number and the geodetic number in this class of graphs do coincide. However, the subsequent characterization of geodetic sets and the algorithm for finding a minimum geodetic set given in [22] (Theorem 6.2 in page 87) are specifically designed for maximal outerplanar graphs and do not generalize to arbitrary biconnected outerplanar graphs.

The main contribution of this paper is a polynomial time algorithm for computing a minimum geodetic set of a biconnected outerplanar graph in particular, and of an outerplanar graph, in general.

The paper is organized as follows. In Section 2 we give definitions and basic properties of outerplanar graphs. In Section 3 we give some fundamental properties of minimum geodetic sets in outerplanar graphs and we introduce the concept of lobe graph. A lobe graph G' is an outerplanar subgraph of a biconnected outerplanar graph G that has a particular simple structure and is used in the design of the algorithm.

In Section 4 we give an algorithm for finding a minimum geodesic set in a biconnected outerplanar graph and prove its correctness. In Section 5 we finally give a polynomial time algorithm for finding a geodetic set of a general outerplanar graph, thus proving that the algorithm for computing a minimum geodetic set has polynomial time complexity for this class of graphs. Based on the fact that, in outerplanar graphs, $I_G[X] = I_G[I_G[X]]$, we obtain also a polynomial time algorithm for computing the hull set and the hull number of a outerplanar graph.

2. Definitions and basic properties of outerplanar graphs

In what follows G will be a finite, connected, undirected, loopless and simple graph. Sets $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. An edge of a graph, which is a set of two vertices, is denoted as uv where u and v are two distinct vertices of G and are called the *endpoints* of uv.

With $|G|$ we will denote $|E(G)|$; however, if S is a set, then $|S|$ still denotes the cardinality of S. The set of vertices adjacent to a vertex v in a graph G is denoted as $N_G(v)$. Given a sequence $(v_0, v_1, \ldots, v_k)$, $k \geq 0$ of distinct vertices, the graph P where

$$V(P) = \{v_0, v_1, \ldots, v_k\} \text{ and } E(P) = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$$

is a *path* of *length* $|P|$. The vertices v_0 and v_k are the *endpoints* of P. We say that P is *between* v_0 and v_k. Given two vertices a, b of a path P we will denote with $P[a, b]$ the *subpath* of P between a and b. If P is a path between two adjacent vertices a and b of length at least two, then the graph C, whose vertex set is $V(C) = V(P)$ and $E(C) = E(P) \cup \{ab\}$, is a *cycle*. Furthermore a cycle C such that $|C| = 3$ is called a *triangle* and C is *odd* or *even* depending on whether $|C|$ is odd or even. As usual, we may define a path or a cycle by simply giving the sequence of its vertices. For example, a path P of length 1 between two vertices u and v will be denoted as $P = (u, v)$.

Let u and v be two vertices of G. A *geodesic* in G is a path between u and v in G of minimum length; the *distance*, $d_G(u, v)$, between u and v in G is the length of a geodesic between u and v in G.

A subset X of $V(G)$ is *geodetic* in G if $I_G[X] = V(G)$ and is a *minimum geodetic set* (MGS) of G if it is geodetic and of minimum cardinality.

A *chord* of a cycle C (or path) is an edge uv of G such that $\{u, v\} \subseteq V(C)$ and $uv \notin E(C)$. A cycle (path) is *chordless* or *induced* if no edge of the graph is a chord of the cycle (path).

A vertex v of G is a *cutpoint* of G if v is a separator of G. A *biconnected component* of a graph G is a maximal subgraph of G having no cutpoints.

An *edge subdivision* is an operation that substitutes in a graph G an edge uv with the two edges uw and wv, where $w \notin V(G)$. Two graphs are *homeomorphic* if both can be obtained from the same graph by a sequence of subdivisions of edges [23].