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Analysis and numerical study of a mixed formulation of a two membranes problem

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ABSTRACT

In this paper, we present a reformulation of a two membranes problem as a mixed formulation problem based on the subdifferential of a continuous function of which the subdifferential leads to the characterization of the coincidence set. The reformulated problem is transformed into a saddle point problem. Then we present an iterative method to solve the problem and we prove the convergence of the approximate solutions to the exact one. Finally, some numerical examples are given.

1. Introduction

The two membranes problem consists of finding the equilibrium position of two uniform elastic membranes fixed at the boundary of a domain Ω of $\mathbb{R}^n$ ($n \geq 1$) and subjected to uniform external forces. We assume that the displacement of the membranes at the boundary is already prescribed. The existence and regularity theory for the problem of two membranes was developed by G. Vergara Caffarelli [1,2] (see also [3]). The first paper deals with uniformly elliptic linear case and the later deals with mean surface curvature equations. The regularity of the free boundary of the coincidence set of the two membranes has been studied in [3]. In [4] the authors reformulate the problem as a system of equations based on the subdifferential of a convex continuous function. In [5], the authors proposed a numerical approximation of the N-membranes problem, where the problem is reformulated, with a change of variables, as a bound constrained quadratic minimization problem. The N-membranes problem has been introduced by M. Chipot and G. Vergara Caffarelli in [6], where the authors studied the existence and regularity of the solution of the problem. Later, in [7,8] the authors studied the existence and regularity of the solutions for quasilinear systems and the problem with non local constraints, respectively.

In this paper, we introduce a new mixed formulation of the two membranes problem, for linear elliptic case, based on appropriate variational inequality of the second kind and the notion of the subdifferential μ of a convex continuous function. This formulation is equivalent to a saddle point problem of which the Lagrange multiplier μ leads to the characterization of the coincidence (or contact) set of the two membranes, which is one of the unknowns of the problem. Then we present an iterative method to solve the problem and we prove the convergence of the approximate solutions. To validate the theoretical study we present some numerical examples.

The paper is structured as follows. In Section 2, we present the mathematical setting of the two membranes problem. In Section 3, we reformulate the problem as a mixed formulation problem and provide a relationship between the subdifferential μ and the coincidence set. Then in Section 4, we give an iterative method based on Uzawa type algorithm to

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approximate the exact solution of the saddle point problem. Section 5 is devoted to prove the convergence of the approximate solution to the exact one. Finally, in Section 6 we give some numerical examples in one and two dimensional spaces.

2. Problem setting

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a smooth boundary $\partial\Omega$, (f_1, f_2) be an element of $L^2(\Omega) \times L^2(\Omega)$ and (g_1, g_2) be an element of $H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, with $g_1 \geq g_2$ on $\partial\Omega$. Set

$$K := \left\{(v_1, v_2) \in L^2(\Omega) \times L^2(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega\right\},$$

and for $i = 1, 2$ we denote

$$H^1_{g_i}(\Omega) := \{v \in H^1(\Omega) : v = g_i \text{ on } \partial\Omega\}.$$

We consider the following problem:

$$\begin{cases} \text{Find } u = (u_1, u_2) \in K \cap \left(H^1_{g_1}(\Omega) \times H^1_{g_2}(\Omega)\right) \text{ such that} \\ \displaystyle\sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla(v_i - u_i)dx + \sum_{i=1}^{2} \int_\Omega f_i(v_i - u_i)dx \geq 0 \quad \forall (v_1, v_2) \in K \cap \left(H^1_{g_1}(\Omega) \times H^1_{g_2}(\Omega)\right). \end{cases} \tag{1}$$

This problem is called the two membranes problem, where the solution (u_1, u_2) describes the displacement of the two membranes.

In this paper we use the same letters g_i, $i = 1, 2$, to denote a function of $H^1(\Omega)$ whose trace on the boundary is the given function g_i, respectively. If we set

$$w_1 := u_1 - g_1 \quad \text{and} \quad w_2 := u_2 - g_2,$$

then problem (1) becomes equivalent to the following problem:

$$\begin{cases} \text{Find } w = (w_1, w_2) \in K_g \cap \left(H^1_0(\Omega) \times H^1_0(\Omega)\right) \text{ such that} \\ \displaystyle\sum_{i=1}^{2} \int_\Omega \nabla(w_i + g_i) \cdot \nabla(v_i - w_i)dx + \sum_{i=1}^{2} \int_\Omega f_i(v_i - w_i)dx \geq 0 \\ \qquad\qquad \forall (v_1, v_2) \in K_g \cap \left(H^1_0(\Omega) \times H^1_0(\Omega)\right), \end{cases} \tag{2}$$

where

$$K_g := \left\{(v_1, v_2) \in L^2(\Omega) \times L^2(\Omega) : v_1 + g_1 \geq v_2 + g_2 \text{ a.e. in } \Omega\right\}.$$

We denote

$$\begin{aligned}
&f := (f_1, f_2). \\
&g := (g_1, g_2). \\
&\left[H^1_0(\Omega)\right]^2 := H^1_0(\Omega) \times H^1_0(\Omega). \\
&\left[L^2(\Omega)\right]^2 := L^2(\Omega) \times L^2(\Omega). \\
&a(u, v) := \int_\Omega \nabla u_1 \cdot \nabla v_1 dx + \int_\Omega \nabla u_2 \cdot \nabla v_2 dx \quad \forall u, v \in \left[H^1(\Omega)\right]^2. \\
&((u, v)) := \int_\Omega u_1 v_1 dx + \int_\Omega u_2 v_2 dx \quad \forall u, v \in \left[L^2(\Omega)\right]^2.
\end{aligned}$$

The bilinear functions $((\cdot,\cdot))$ and $a(\cdot,\cdot)$ are inner products of $\left[L^2(\Omega)\right]^2$ and $\left[H^1_0(\Omega)\right]^2$, respectively. Then problem (2) can be written in the following form:

$$\begin{cases} \text{Find } w \in K_g \cap \left[H^1_0(\Omega)\right]^2 \text{ such that} \\ a(w + g, v - w) + ((f, v - w)) \geq 0 \quad \forall v \in K_g \cap \left[H^1_0(\Omega)\right]^2. \end{cases} \tag{3}$$

This problem admits a unique solution. Indeed, it is easy to see that $K_g \cap \left[H^1_0(\Omega)\right]^2$ is a non empty closed convex set of $\left[H^1_0(\Omega)\right]^2$. The application $h : \left[H^1_0(\Omega)\right]^2 \to \mathbb{R}$ defined by $h(v) := a(g, v) + ((f, v))$ is a continuous linear form on $\left[H^1_0(\Omega)\right]^2$. Moreover the bilinear form $a(\cdot,\cdot)$ defines an inner product on $\left[H^1_0(\Omega)\right]^2$, then it is continuous and coercive on $\left[H^1_0(\Omega)\right]^2$. Hence problem (3) admits a unique solution (see [9,3]).

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